EE 505

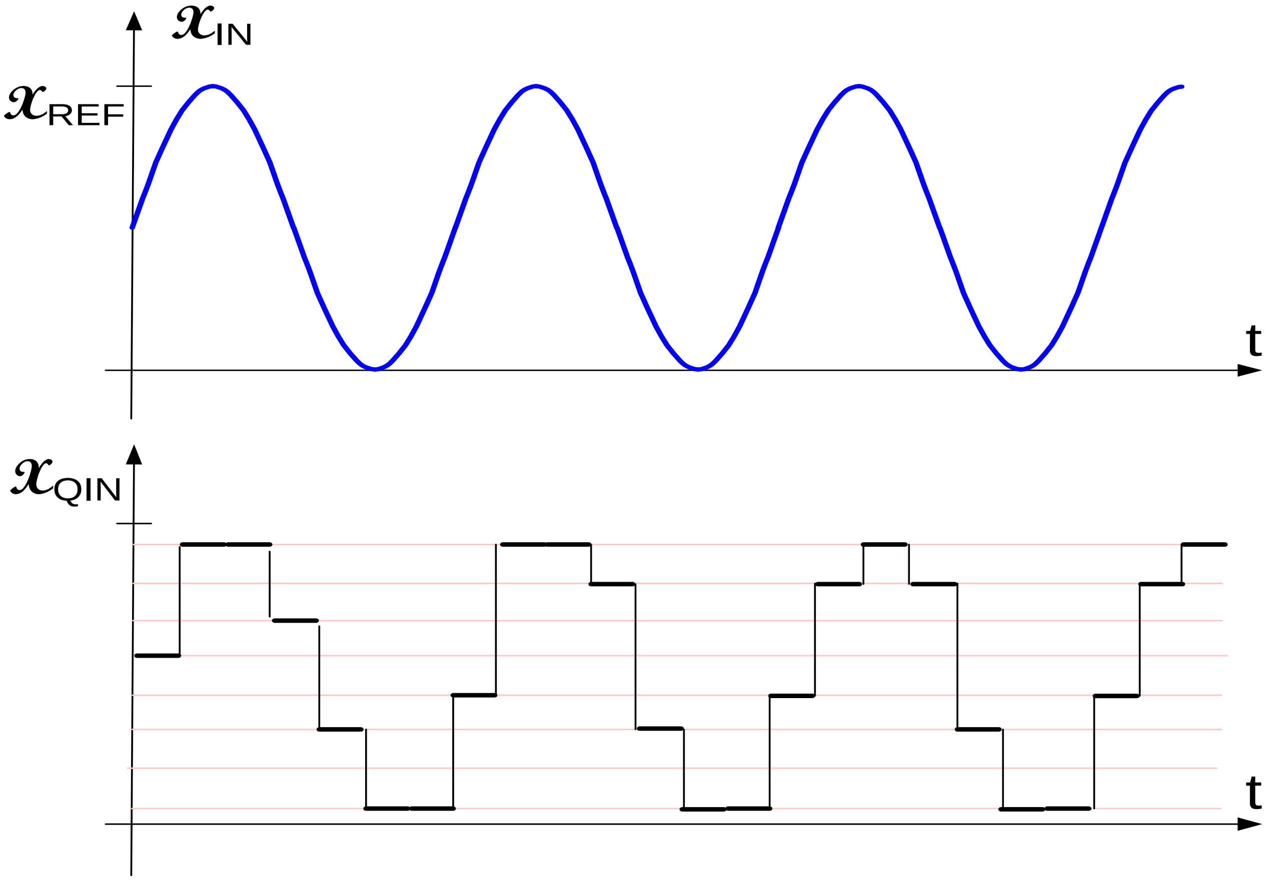
Lecture 5

Spectral Characterization

Review from Last Lecture

Quantization Noise in ADC

How does the SNR change if the input is a sinusoid that goes from 0 to $\mathcal{X}_{REF}$ centered at $\mathcal{X}_{REF}/2$?



Time and Amplitude Quantized Waveform

ENOB based upon Quantization Noise

$$SNR = 6.02\,n + 1.76$$

Solving for n, obtain

$$ENOB = \frac{SNR_{dB} - 1.76}{6.02}$$

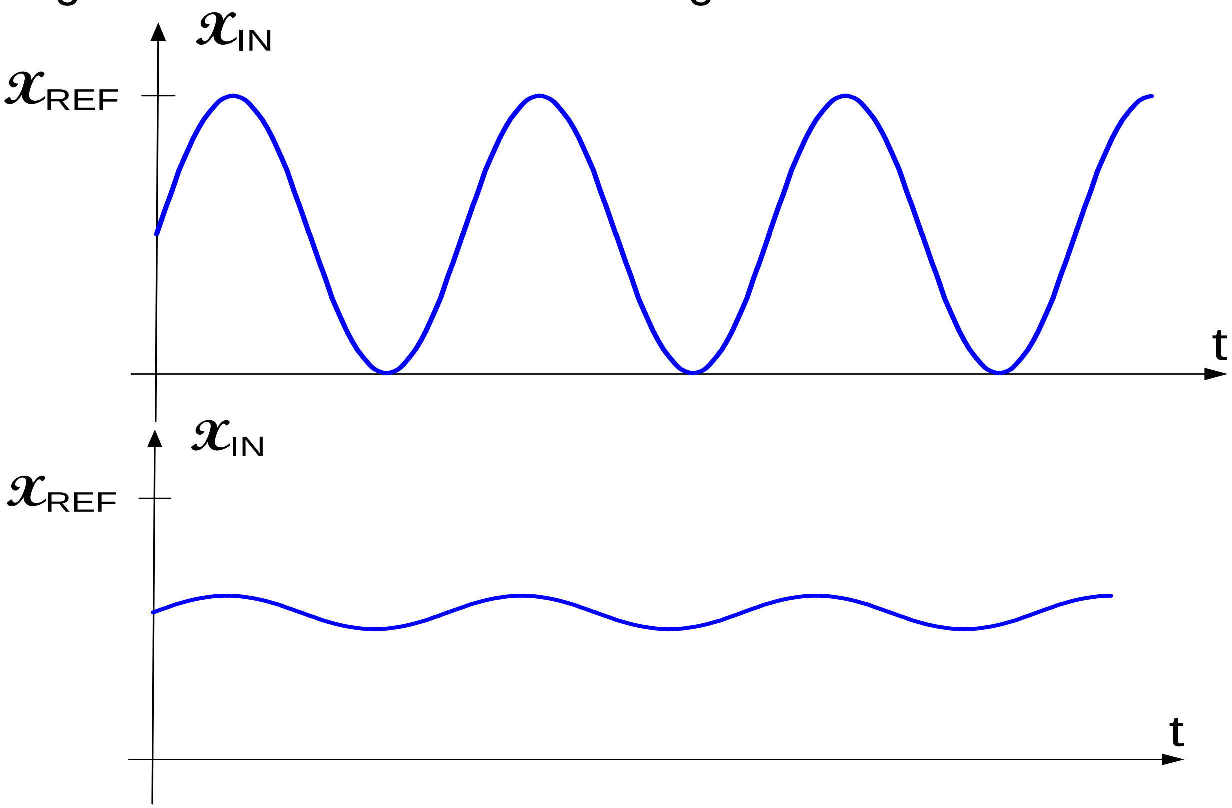
Note: could have used the SNR_{dB} for a triangle input and would have obtained the expression

$$ENOB = \frac{SNR_{dB}}{6.02}$$

But the earlier expression is more widely used when specifying the ENOB based upon the noise level present in a data converter

Quantization Noise

Effects of quantization noise can be very significant, even at high resolution, when signals are not of maximum magnitude



Quantization noise remains constant but signal level is reduced

The desire to use a data converter at a small fraction of full range is one of the major reasons high resolution is required in many applications

Review from Last Lecture

ENOB Summary

Resolution:

$$\text{ENOB} = \frac{\log_{10} N_{ACT}}{\log_{10} 2} = \log_2 N_{ACT}$$

INL:

$$\text{ENOB} = n_R - \log_2(\upsilon) - 1$$

n_R specified res, v INL in LSB

$$\text{ENOB} = -\log_2(\text{INL}_{REF}) - 1$$

INL_{REF} INL rel to X_{REF}

DNL:

HW problem

Quantization noise:

$$\text{ENOB} = \frac{\text{SNR}_{dB}}{6.02}$$

rel to triangle/sawtooth

$$\text{ENOB} = \frac{\text{SNR}_{dB} - 1.76}{6.02}$$

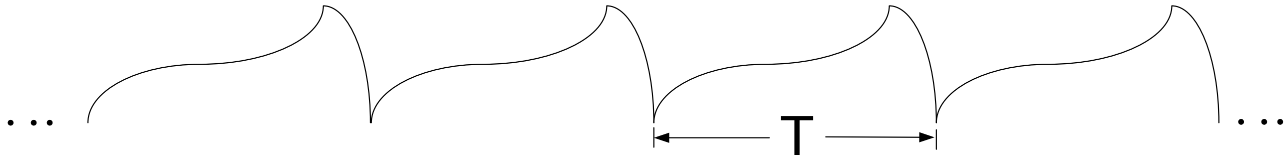
rel to sinusoid

Most widely used for static characteristics

Additional ENOB will be introduced when discussing dynamic characteristics

Review from Last Lecture

Spectral Analysis



If f(t) is periodic

$$f(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin\left(k\omega t + \theta_k\right)$$

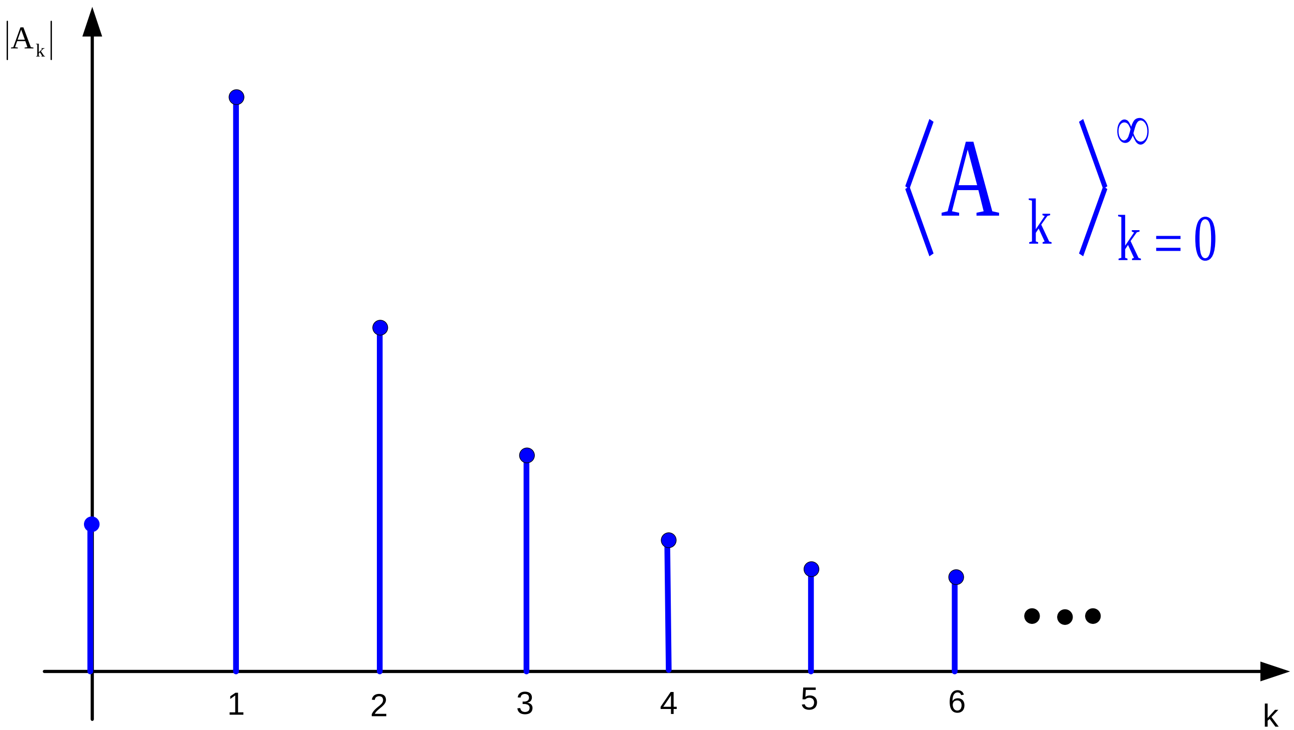
alternately

$$f(t) = A_0 + \sum_{k=1}^{\infty} a_k \sin\left(k\omega t\right) + \sum_{k=1}^{\infty} b_k \cos\left(k\omega t\right) \qquad \omega = \frac{2\pi}{T}$$

$$A_k = \sqrt{a_k^2 + b_k^2}$$

Termed the Fourier Series Representation of f(t)

Distortion Analysis

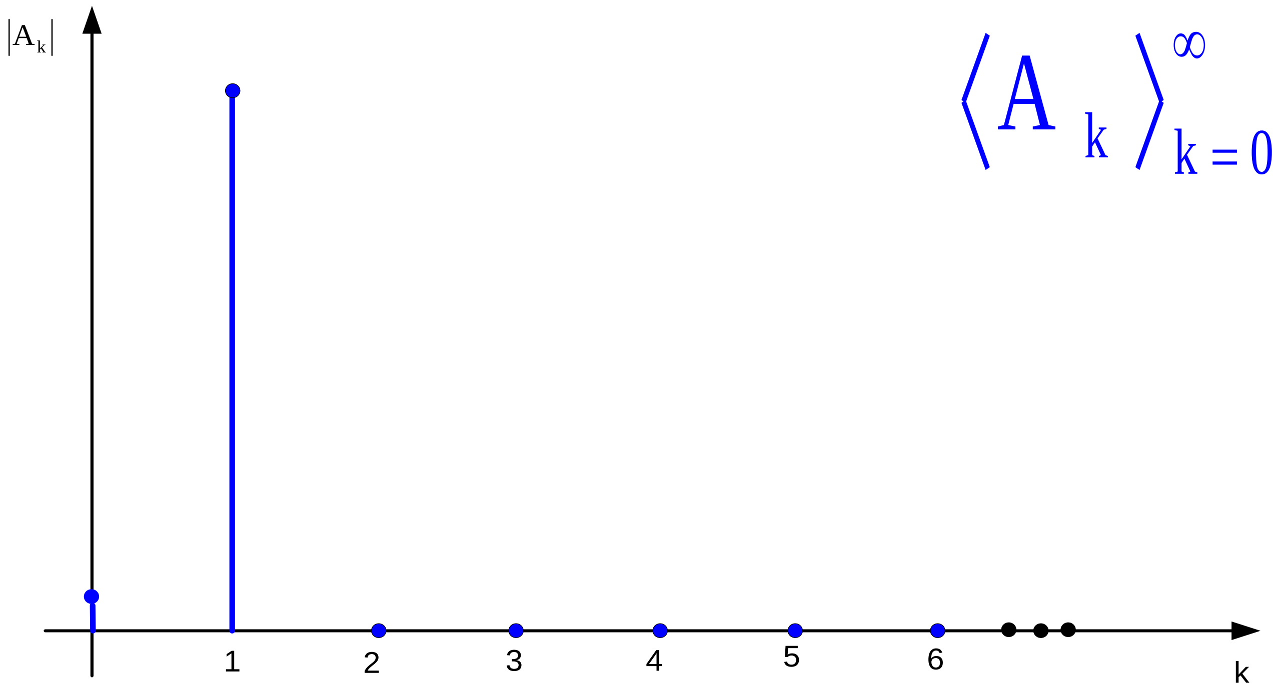


- Distortion analysis is another approach for assessing linearity
- Often termed the DFT coefficients (will show later)
- Spectral lines, not a continuous function

A_1 is termed the fundamental (when input is sinusoid or periodic)

A_k is termed the kth harmonic (when input is sinusoid or periodic)

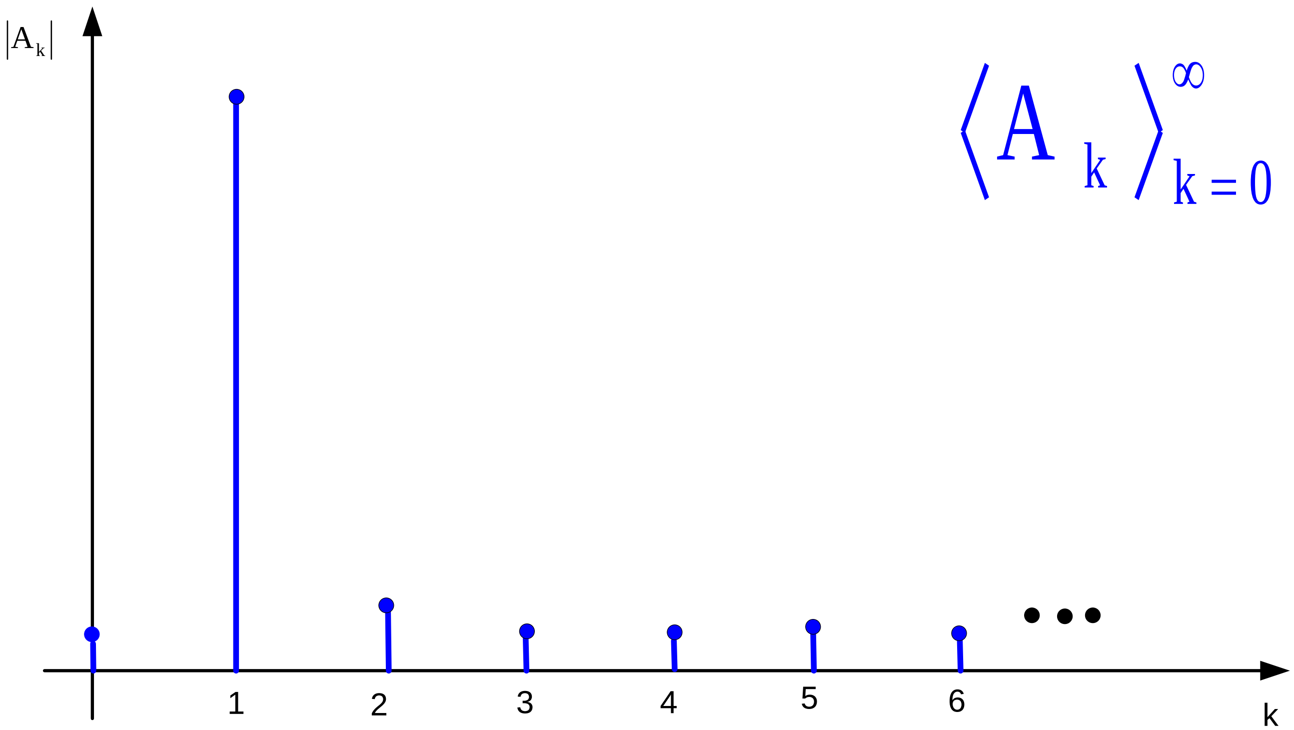
Distortion Analysis



$$\langle A_k \rangle_{k=0}^{\infty}$$

Often ideal response will have only fundamental present and all remaining spectral terms will vanish

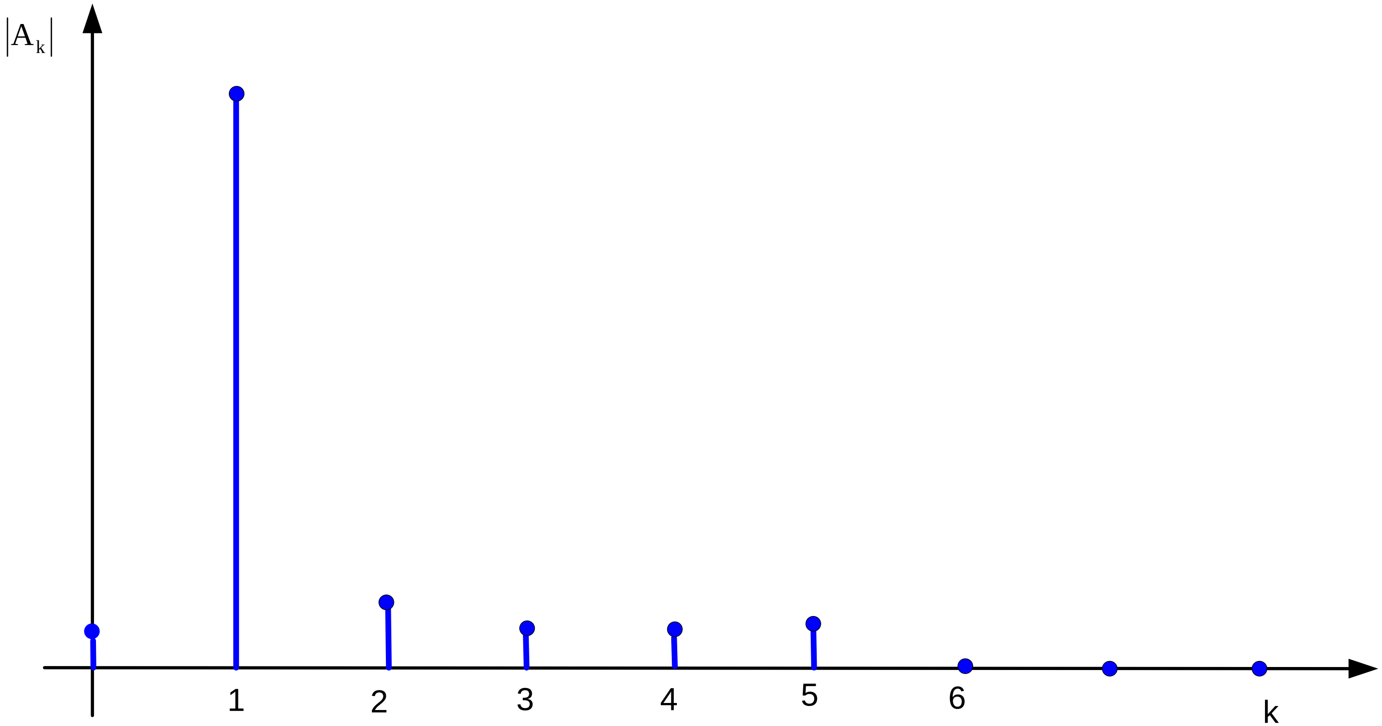
Distortion Analysis



For a low distortion signal, the 2nd and higher harmonics are generally much smaller than the fundamental

The magnitude of the harmonics generally decrease rapidly with k for low distortion signals

Distortion Analysis



Assume f(t) is periodic with period $T = \frac{1}{f}$

f(t) is band-limited to frequency $2\pi f k_X$ if A_k=0 for all $k>k_x$

where $\langle A_k \rangle_{k=0}^{\infty}$ are the Fourier series coefficients of f(t)

Distortion Analysis

Total Harmonic Distortion, THD

$$\mathrm{THD} = \frac{\text{RMS voltage in harmonics}}{\text{RMS voltage of fundamental}}$$

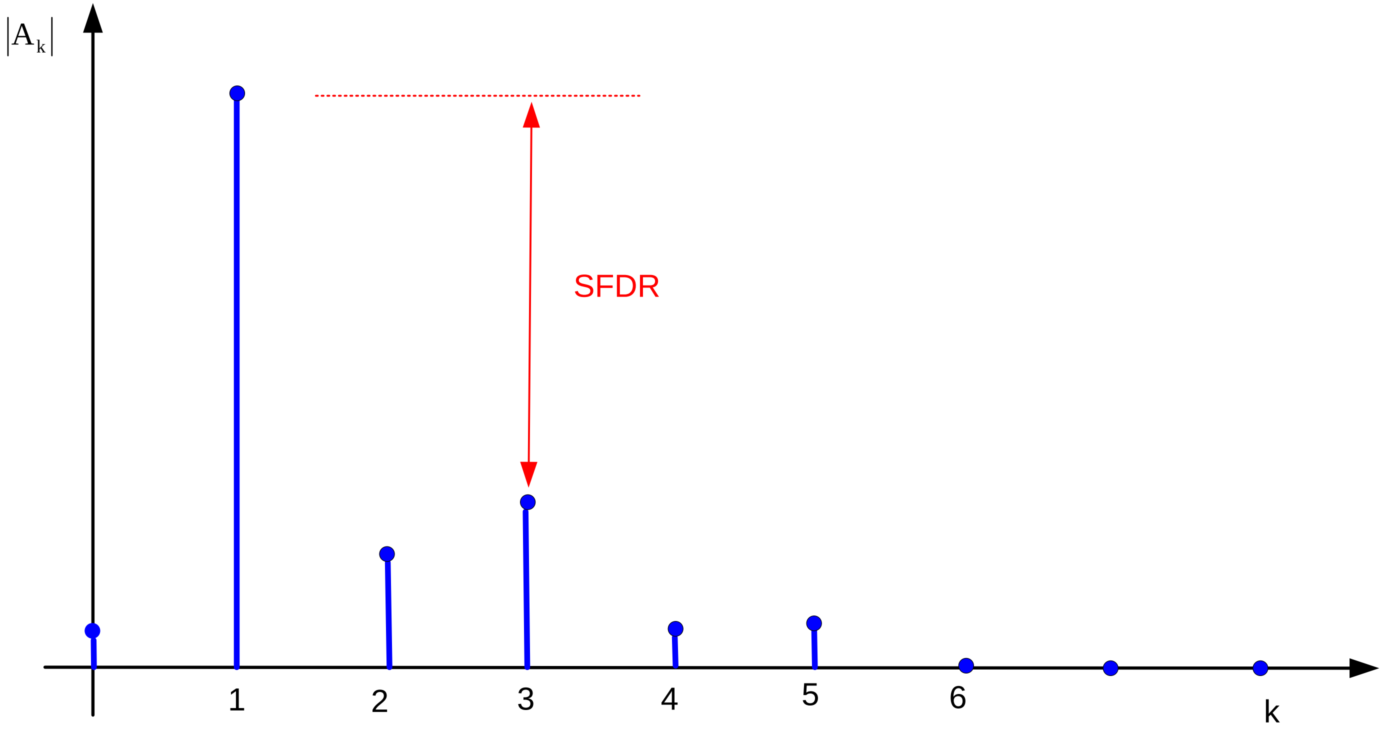
$$\mathrm{THD} = \frac{\sqrt{\left(\frac{A_2}{\sqrt{2}}\right)^2 + \left(\frac{A_3}{\sqrt{2}}\right)^2 + \left(\frac{A_4}{\sqrt{2}}\right)^2 + \ldots}}{\frac{A_1}{\sqrt{2}}}$$

$$\mathrm{THD} = \frac{\sqrt{\sum_{k=2}^{\infty} A_k^2}}{A_1}$$

Distortion Analysis

Spurious Free Dynamic Range, SFDR

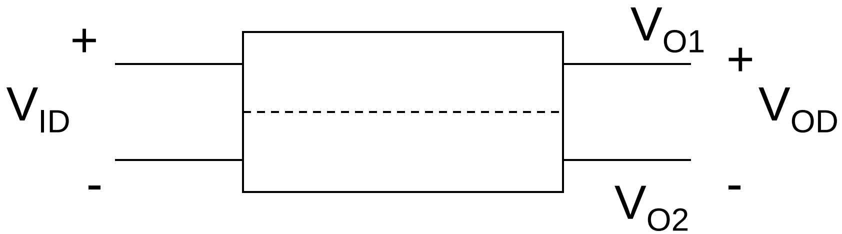
The SFDR is the difference between the fundamental and the largest harmonic



SFDR and THD are usually determined by either the second or third harmonic

Distortion Analysis

Theorem: In a fully differential symmetric circuit, all even-order terms are absent in the Taylor's series output for symmetric differential sinusoidal excitations !



Proof: Expanding $g(V_{ID})$ in a Taylor's series around $V_{ID}=0$, we obtain

$$V_{01} = g\left(V_{ID}\right) = \sum_{k=0}^{\infty} h_k V_{ID}^k$$

$$V_{02} = g\left(\text{-}V_{ID}\right) = \sum_{k=0}^{\infty} h_k \left(\text{-}V_{ID}\right)^k$$

$$V_{OD} = V_{01} - V_{02} = \sum_{k=0}^{\infty} h_k \left(V_{ID}\right)^k - \sum_{k=0}^{\infty} h_k \left(\text{-}V_{ID}\right)^k$$

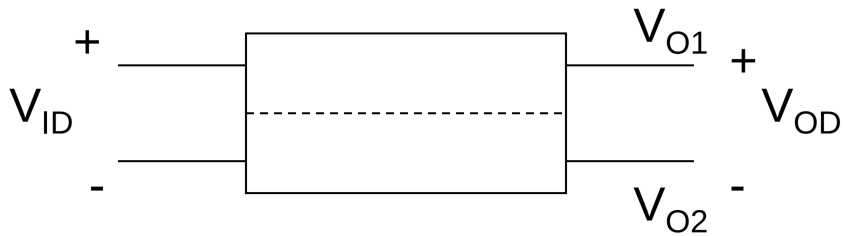
$$V_{OD} = \sum_{k=0}^{\infty} h_k \left[\left(V_{ID}\right)^k - \left(\text{-}V_{ID}\right)^k\right]$$

$$V_{OD} = \sum_{k=0}^{\infty} h_k \left[\left(V_{ID}\right)^k - \left(-1\right)^k \left(V_{ID}\right)^k\right]$$

When k is even, the corresponding term in [ ] vanishes

Distortion Analysis

Theorem: In a fully differential symmetric circuit, all even harmonics are absent in the differential output for symmetric differential sinusoidal excitations !



Proof: From prev theorem for $V_{ID} = \alpha \sin(\omega t)$ $\quad V_{OD} = \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \beta_k \sin^k(\omega t)$

Recall:

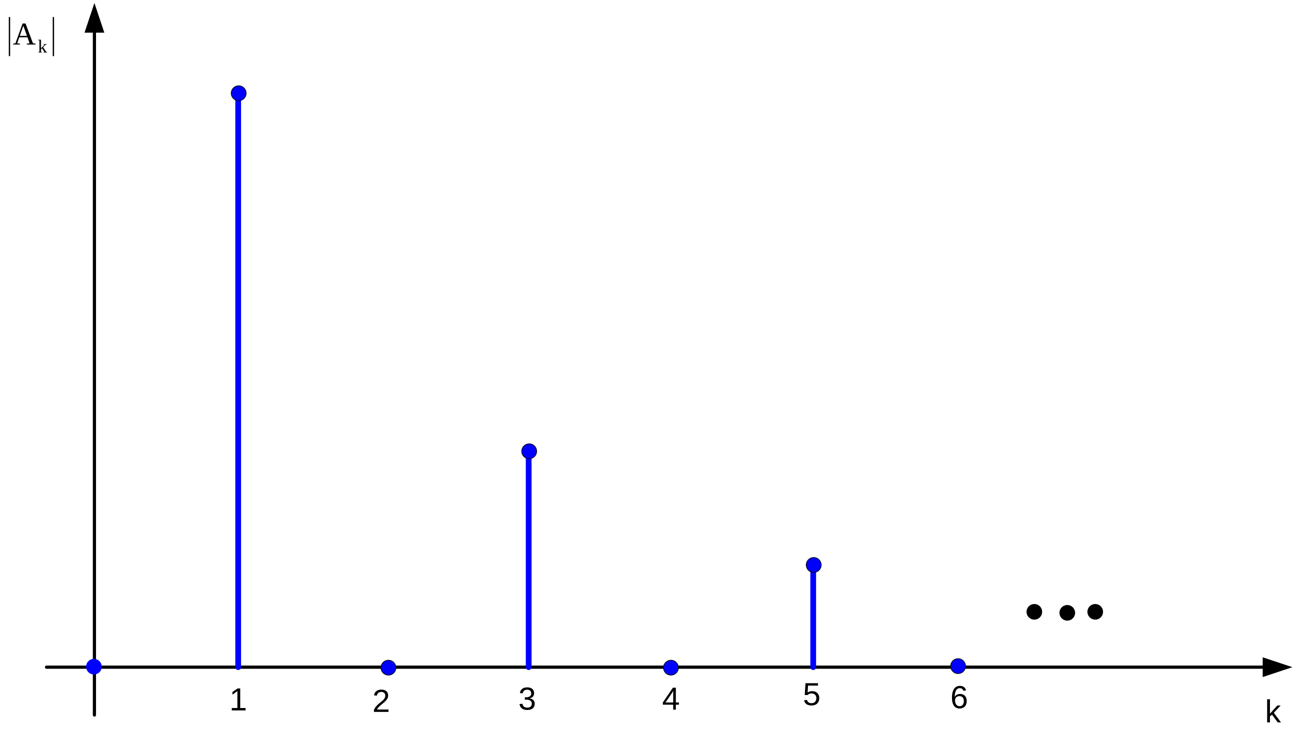
$$\sin^n(x) = \begin{cases} \sum_{k=0}^{\frac{n-1}{2}} h_k \sin\left((n-2k)x\right) & \textit{for n odd} \\ \\ \sum_{k=0}^{\frac{n-2}{2}} g_k \sin\left((n-2k)x + \theta_k\right) & \textit{for n even} \end{cases}$$

where h_k, g_k, and θ_k are constants

That is, odd powers of $\sin^n(x)$ have only odd harmonics present and even powers have only even harmonics present

Distortion Analysis

In a fully differential symmetric circuit, all even harmonics are absent in the differential output !



Distortion Analysis

Consider a time-periodic function g(t)

How are spectral magnitude components of g determined?

By integral

$$A_k = \frac{1}{\omega T}\left(\int_{t_1}^{t_1+T} g(t)e^{-jk\omega t}dt + \int_{t_1}^{t_1+T} g(t)e^{jk\omega t}dt \right)$$

or

$$a_k = \frac{2}{\omega T}\int_{t_1}^{t_1+T} g(t)\sin(kt\omega)dt \qquad b_k = \frac{2}{\omega T}\int_{t_1}^{t_1+T} g(t)\cos(kt\omega)dt$$

Integral is very time consuming, particularly if large number of components are required

By DFT (with some restrictions that will be discussed)

By FFT (special computational method for obtaining DFT)

Frequency Representations of Time-Domain Waveforms

- DFT Discrete Fourier Transform
- DTFT Discrete Time Fourier Transform
- DFS Discrete Fourier Series
- FT Fourier Transform
- FS Fourier Series
- FFT Fast Fourier Transform
- L Laplace Transform

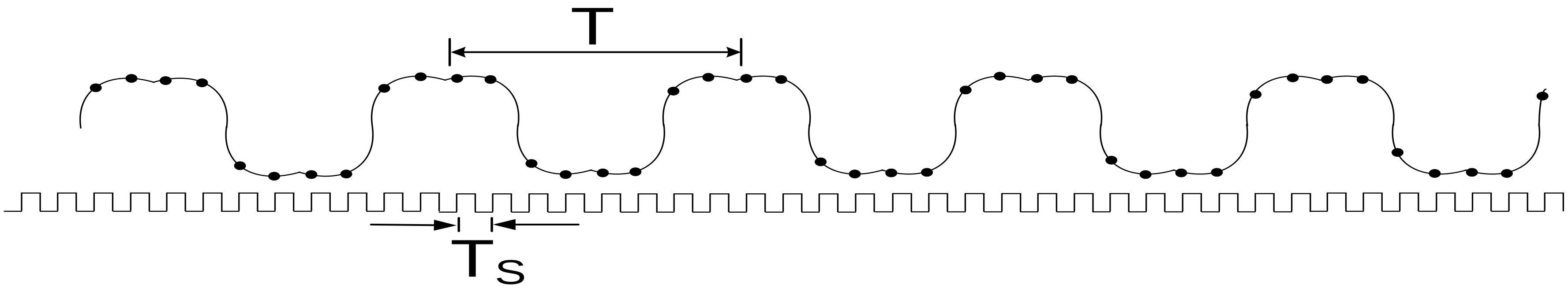
FS ⟺ DFS=DFT

DFT used to characterize linearity of data converters (and many other linear systems)

FFT is computationally efficient algorithm for calculating DFT

Distortion Analysis

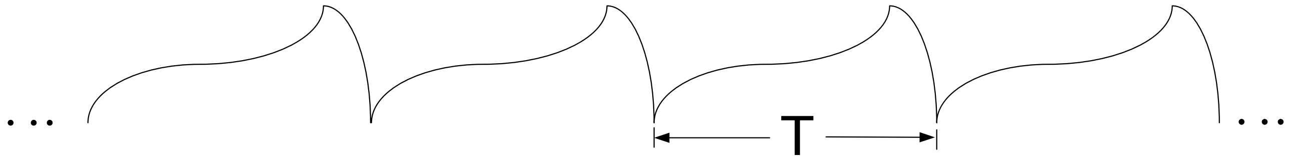
How are spectral components determined?



Consider sampling g(t) at uniformly spaced points in time T_S seconds apart

This gives a sequence of samples $\left\langle g\left(kT_s\right)\right\rangle_{k=1}^{N}$

Distortion Analysis



Consider a function g(t) that is periodic with period T

$$g(t) = A_0 + \sum_{k=1}^{\infty} A_k \sin(k\omega t + \theta_k)$$

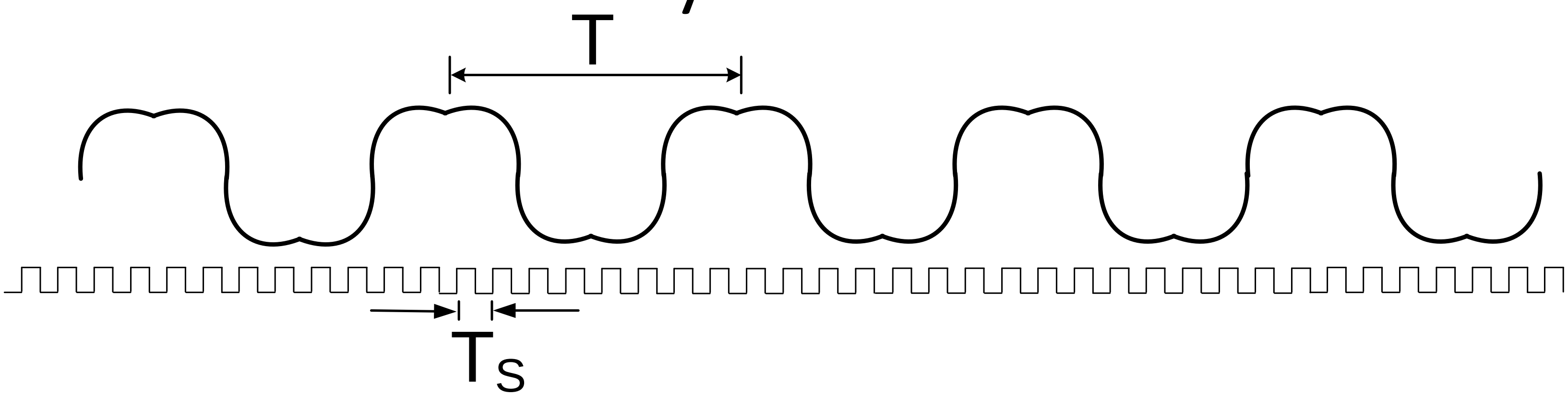
$$\omega = 2\pi f = \frac{2\pi}{T}$$

Band-limited Periodic Functions

Definition: A periodic function of frequency f is band limited to a frequency f_{max} if A_k=0 for all $k > \frac{f_{max}}{f}$

(here $>$ and $\geq$ are not synonymous since in discrete rather than continuous domains)

Distortion Analysis



NOTATION:

T:	Period of Excitation
T_S:	Sampling Period
N_P:	Number of periods over which samples are taken
N:	Total number of samples

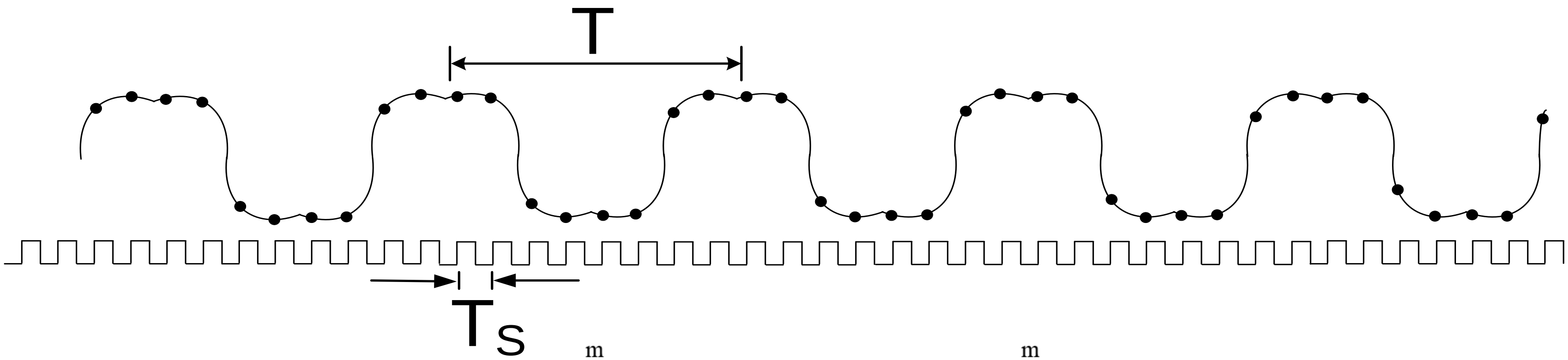
$$N_P = \frac{NT_S}{T}$$

Note: N_P is not an integer unless a specific relationship exists between N, T_S and T

$$h = \mathrm{Int}\left(\left[\frac{N}{2} - 1\right]\frac{1}{N_P}\right)$$

Note: The function Int(x) is the integer part of x

Distortion Analysis



Band-limited signal: $X(t) = A_0 + \sum_{k=1}^{m} A_k \sin(k\omega t + \theta_k) = A_0 + \sum_{k=1}^{m} A_k \sin(k \bullet f_x \bullet 2\pi \bullet t + \theta_k), \quad A_m \neq 0$

Fundamental frequency: f_X Nyquist rate: $2mf_X$

Often interested in how nonlinearity affects a sinusoidal input signal

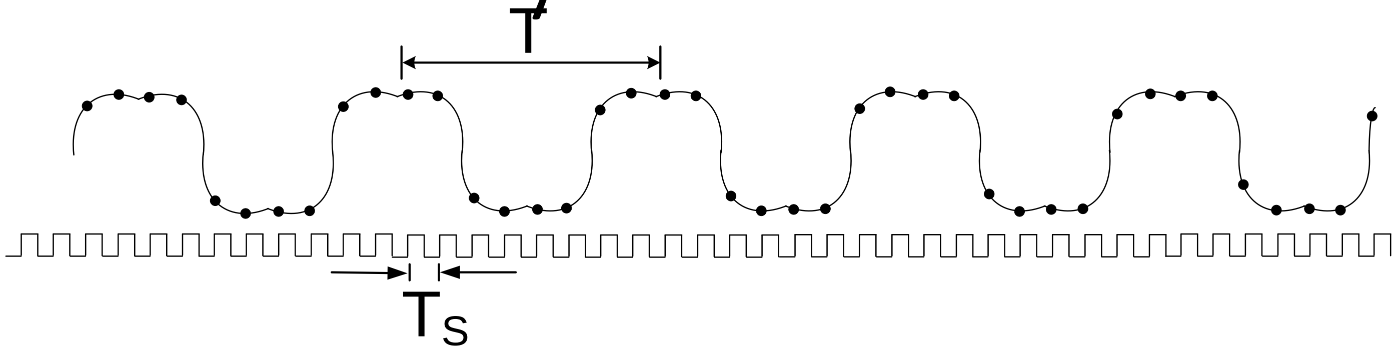
Completely characterized by the Fourier Series Coefficients $<A_k>$ of output

The Fourier Series Coefficients $<A_k>$ are another useful measure of linearity

THEOREM (conceptual) : If a band-limited periodic signal is sampled at a rate that exceeds the Nyquist rate, then the Fourier Series coefficients can be directly obtained from the DFT of a sampled sequence.

$$\langle x(kT_S)\rangle_{k=0}^{N-1} \Longleftrightarrow \langle X(k)\rangle_{k=0}^{N-1}$$

Distortion Analysis



THEOREM: Consider a periodic signal with period T=1/f and sampling period T_S=1/f_S. If N_P is an integer, x(t) is band limited to f_{MAX}, and f_s>2f_{max}, then

$$\left|A_m\right| = \frac{2}{N}\left|X\left(mN_P + 1\right)\right| \qquad 0 \le m \le h-1$$

and $X(k) = 0$ for all k not defined above

where $\left\langle X(k)\right\rangle_{k=0}^{N-1}$ is the DFT of the sequence $\left\langle x(kT_S)\right\rangle_{k=0}^{N-1}$

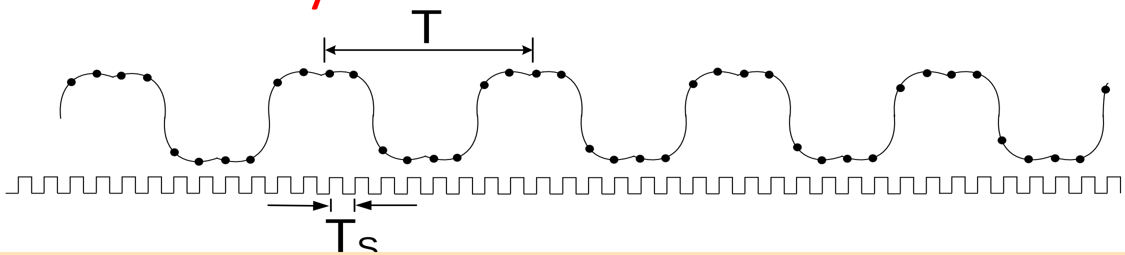
N=number of samples, N_P is the number of periods, and $h = \text{Int}\left(\frac{f_{MAX}}{f} - \frac{1}{N_P}\right)$

N_P an integer means $N_P = N\frac{T_S}{T}$ is an integer

Spectral components of interest are $|A_m|$, m=0….h-1

Key Theorem central to Spectral Analysis that is widely used !!! and often “abused”

Why is this a Key Theorem?



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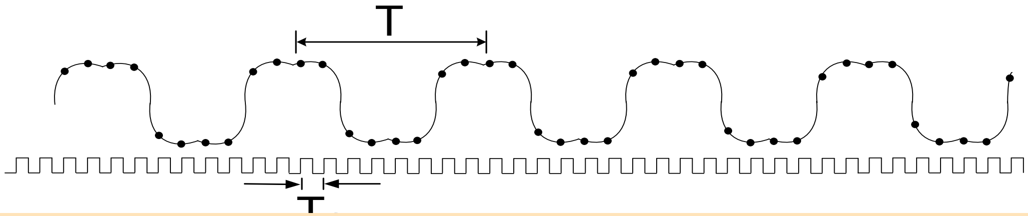
N=number of samples, N_P is the number of periods, and $h = \mathrm{Int}\left(\frac{f_{MAX}}{f} - \frac{1}{N_P}\right)$

- DFT requires dramatically less computation time than the integrals for obtaining Fourier Series coefficients
- Can easily determine the sampling rate (often termed the Nyquist rate) to satisfy the band limited part of the theorem if signal is band limted
- If "signal" is output of a system (e.g. ADC or DAC), f_{MAX} is independent of f

How is this theorem abused?



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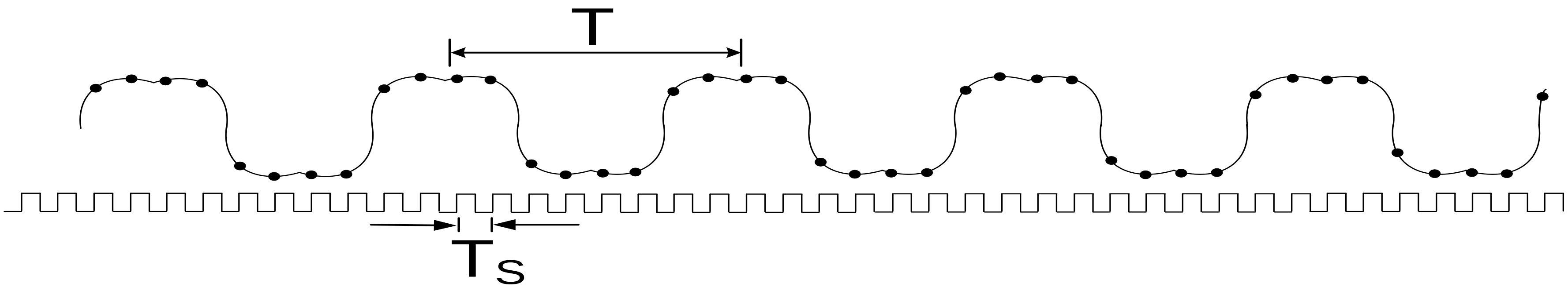
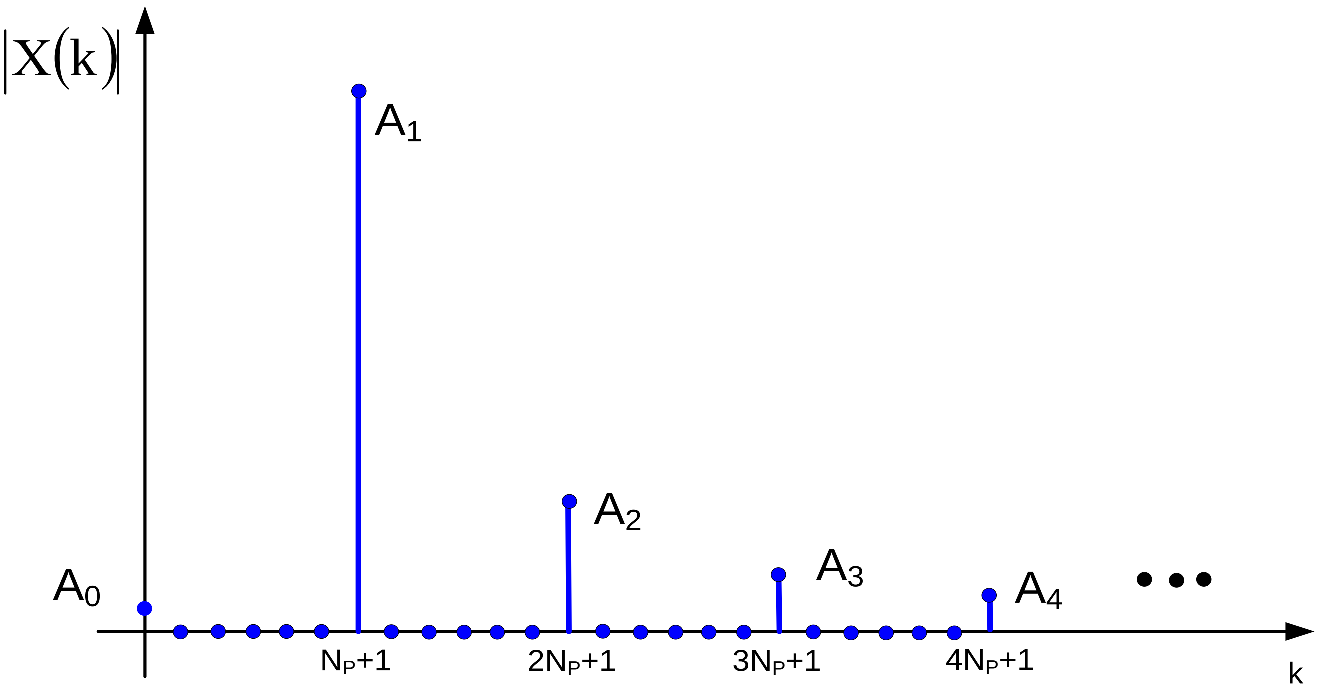
N=number of samples, N_P is the number of periods, and $h = \text{Int}\left(\frac{f_{MAX}}{f} - \frac{1}{N_P}\right)$

- Much evidence of engineers attempting to use the theorem when N_P is not an integer
- Challenging to have N_P an integer in practical applications
- Dramatic errors can result if there are not exactly an integer number of periods in the sampling window

Distortion Analysis



If the hypothesis of the theorem are satisfied, we thus have



Distortion Analysis

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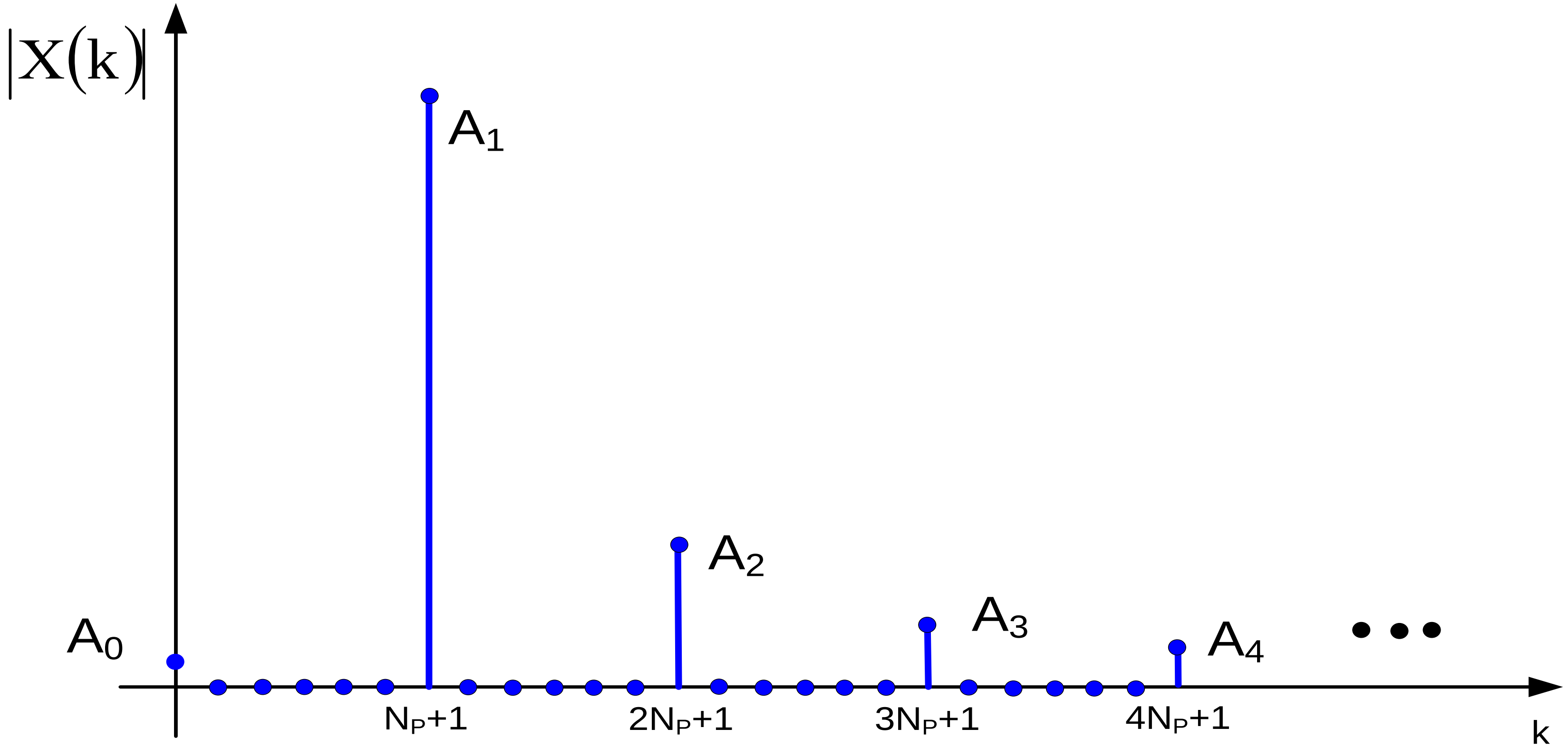
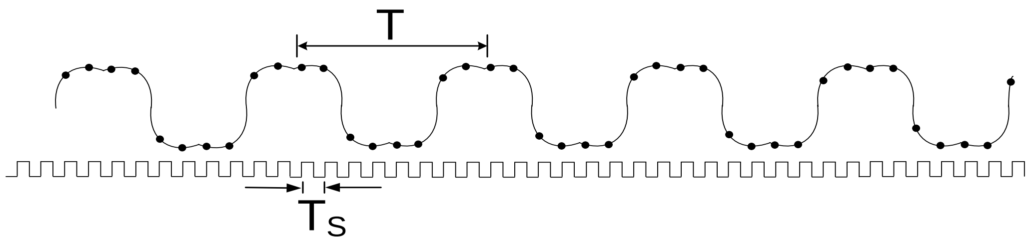


FFT is a computationally efficient way of calculating the DFT, particularly when N is a power of 2



THEOREM: Consider a periodic signal with period T=1/f and sampling period $T_S=1/f_S$. If N_P is an integer, x(t) is band limited to f_{MAX}, and $f_s>2f_{max}$, then

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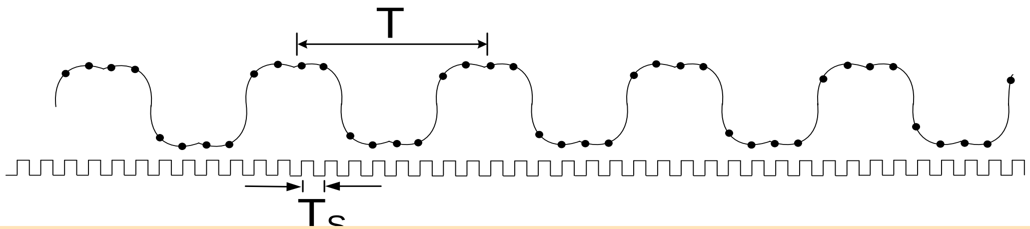
N=number of samples, N_P is the number of periods, and $h = \mathrm{Int}\left(\frac{f_{MAX}}{f} - \frac{1}{N_P}\right)$

Note: Band limited part of the hypothesis is equivalent to stating that the sampling rate is greater than the Nyquist rate of x(t)



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N=number of samples, N_P is the number of periods, and $h = \text{Int}\left(\frac{f_{MAX}}{f} - \frac{1}{N_P}\right)$

Question: Why are we limiting our inputs to periodic signals?

We are looking for metrics for characterizing the linearity of a data converter

FFT Examples

Recall the theorem that provided for the relationship between the DFT terms and the Fourier Series Coefficients required integral number of periods and sampling rate to exceed the Nyquist rate

1. The sampling window be an integral number of periods

2. $$N > \frac{2\, f_{max}}{f_{SIG}} N_P$$

from:

$$f_{SAMP} > 2f_{max}$$

$$NT_{SAMP} = N_P T_{SIG}$$

$$Nf_{SIG} = N_P f_{SAMP}$$

$$f_{MAX} \leq \frac{f_{SIG}}{2} \bullet \left[\frac{N}{N_P} \right]$$

Considerations for Spectral Characterization

- Tool Validation
- FFT Length
- Importance of Satisfying Hypothesis
- Windowing

Considerations for Spectral Characterization

- Tool Validation (MATLAB)

- FFT Length

- Importance of Satisfying Hypothesis

- Windowing

FFT Examples

Recall the theorem that provided for the relationship between the DFT terms and the Fourier Series Coefficients required

1. The sampling window must be an integral number of periods
2. $$N > \frac{2\,f_{max}}{f_{SIG}} N_P$$

Example

WLOG assume f_{SIG}=50Hz

$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

- Consider N_P=11 N=512
- Infinite Resolution

Example

WLOG assume f_{SIG}=50Hz

$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

$f_{MAX\text{-}ACT}$=100Hz

Consider N_P=11 N=512 Infinite Resolution

$$f_{MAX} = \frac{f_{SIG}}{2} \bullet \left[\frac{N}{N_P}\right] = \frac{50}{2} \bullet \frac{512}{11} = 1.164\ \text{KHz}$$

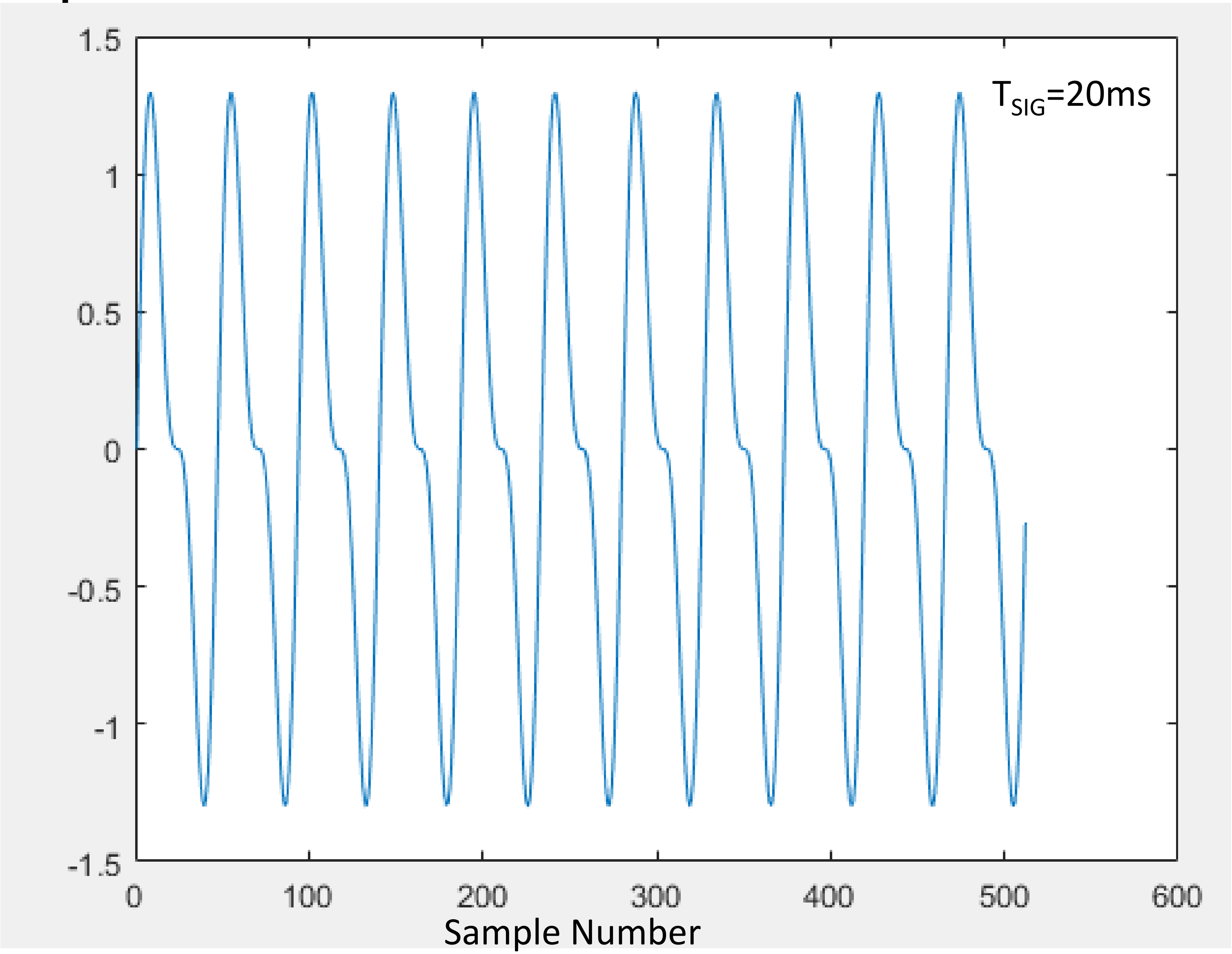
$$f_{MAX\text{-}ACT} << f_{MAX}$$

$$f_{SAMPLE} = \frac{1}{T_{SAMPLE}} = \frac{1}{\left(\frac{N_P \bullet T_{SIG}}{N}\right)} = \left[\frac{N}{N_P}\right] f_{SIG} = 2f_{MAX} = 2.327\ldots \quad \text{KHz}$$

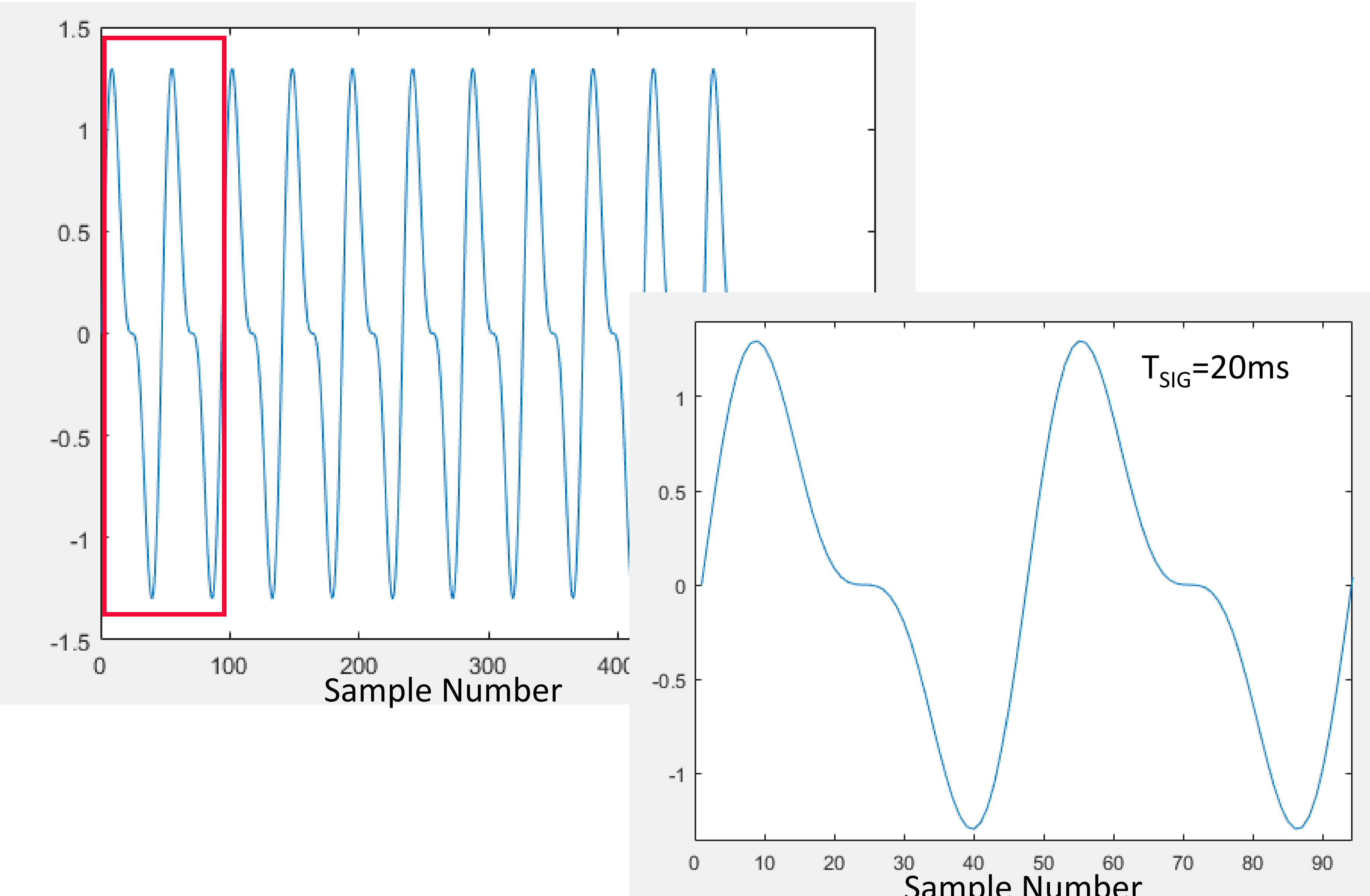
Recall $20\log_{10}(1.0)=0.0000000$

Recall $20\log_{10}(0.5)=-6.0205999$

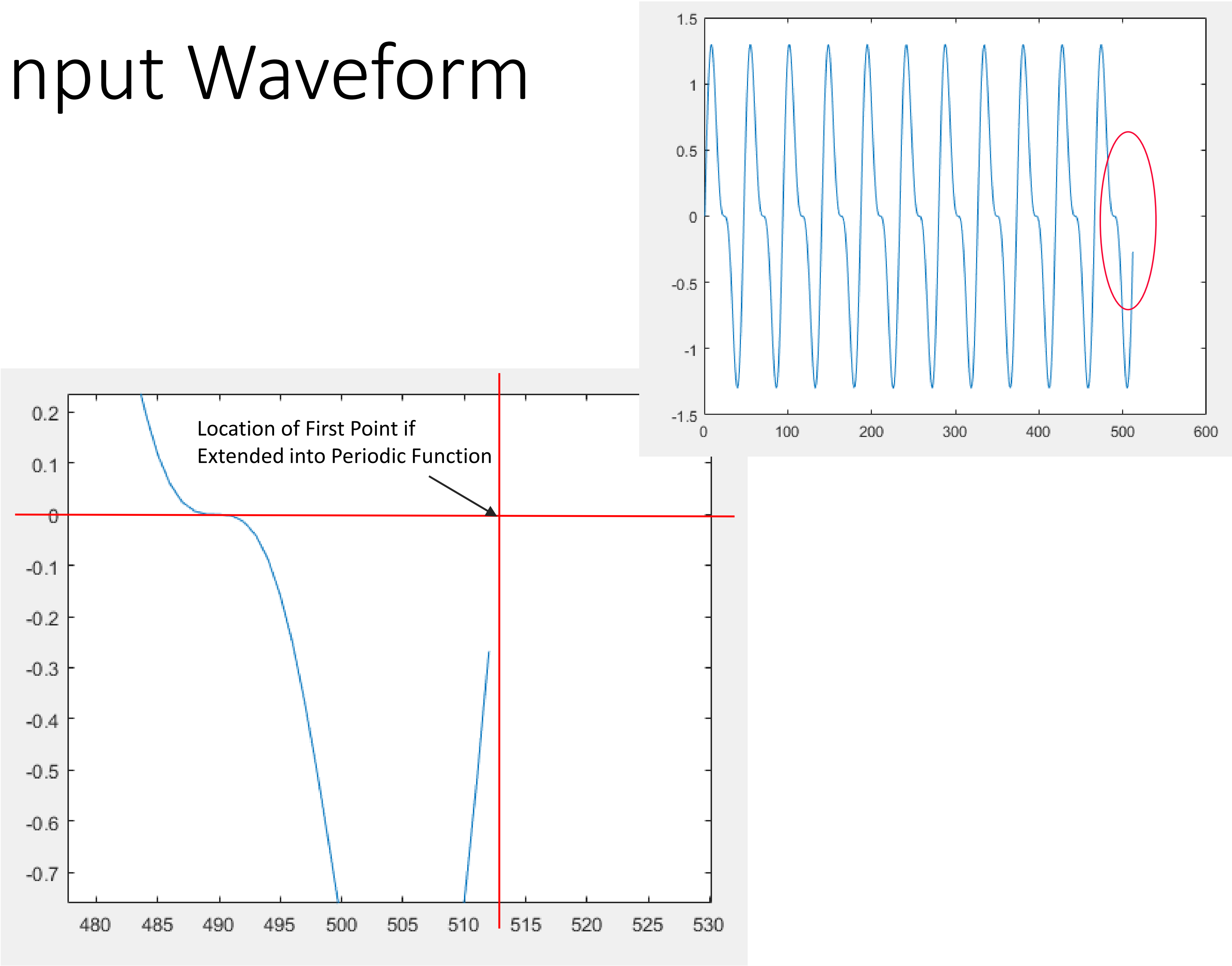
Input Waveform



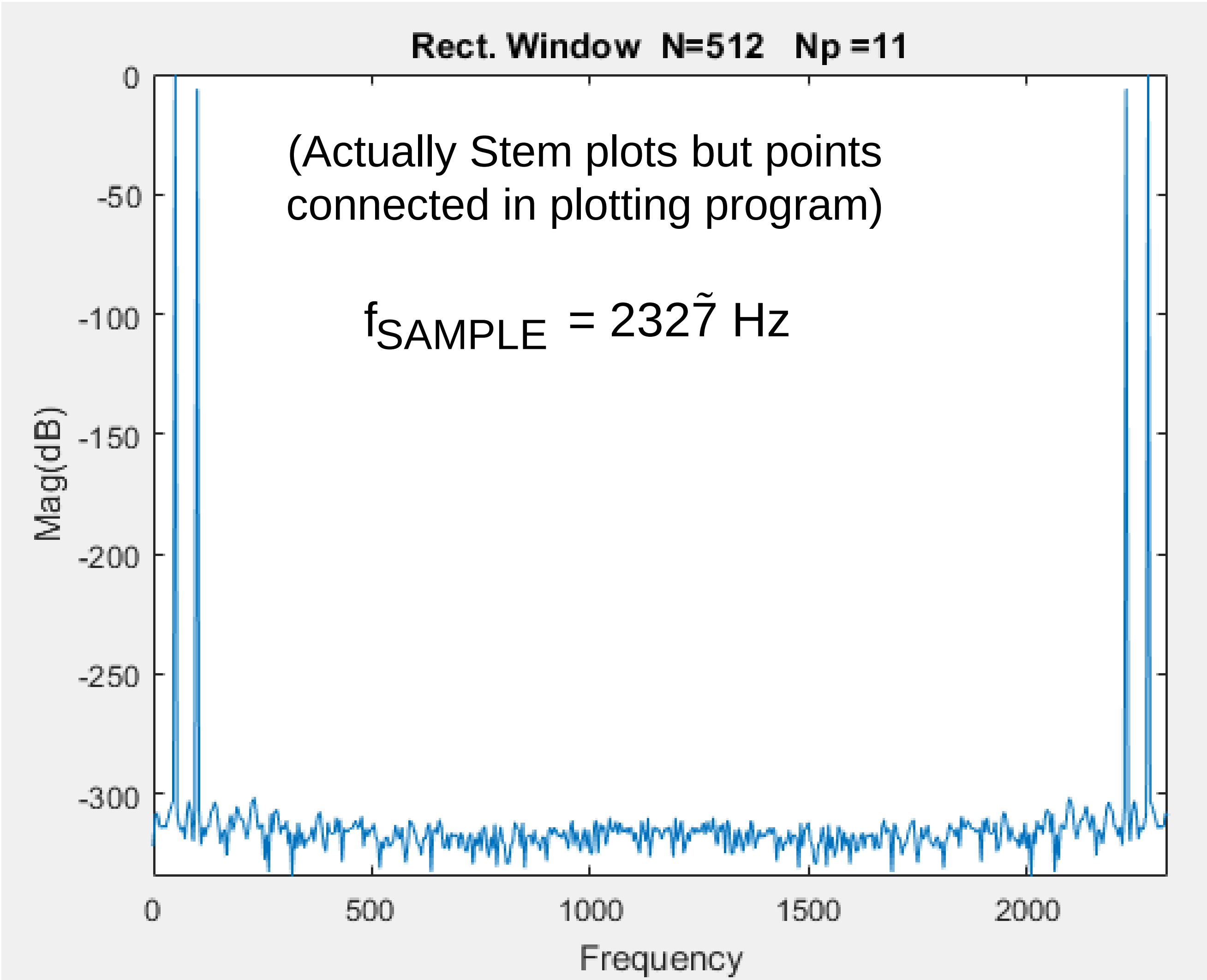
Input Waveform



Input Waveform

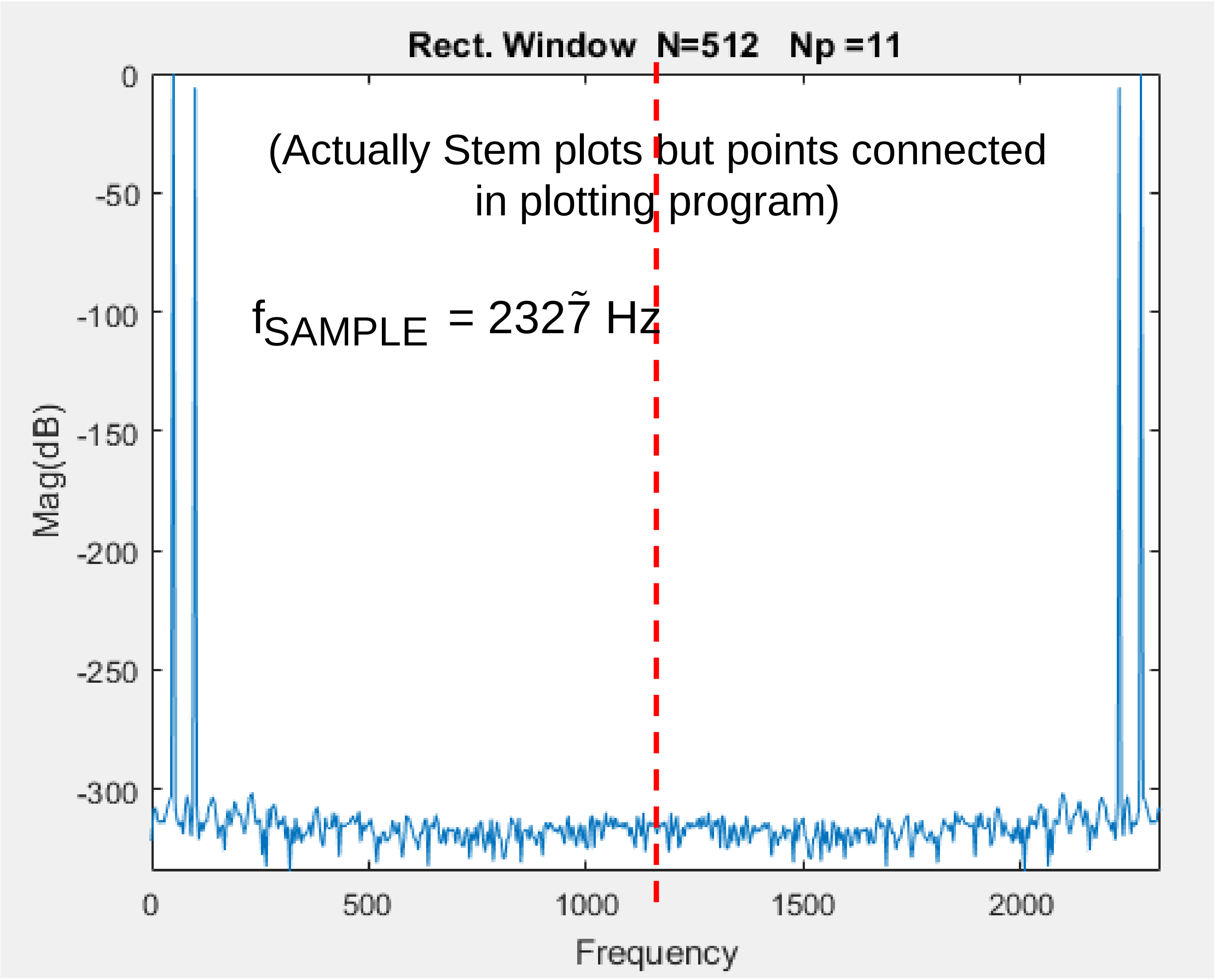


Spectral Response from MATLAB (expressed in dB)



(Horizontal axis is the "Index" axis (k) but converted to frequency) $f_{AXIS} = f_{SIGNAL} \frac{k-1}{N_P}$

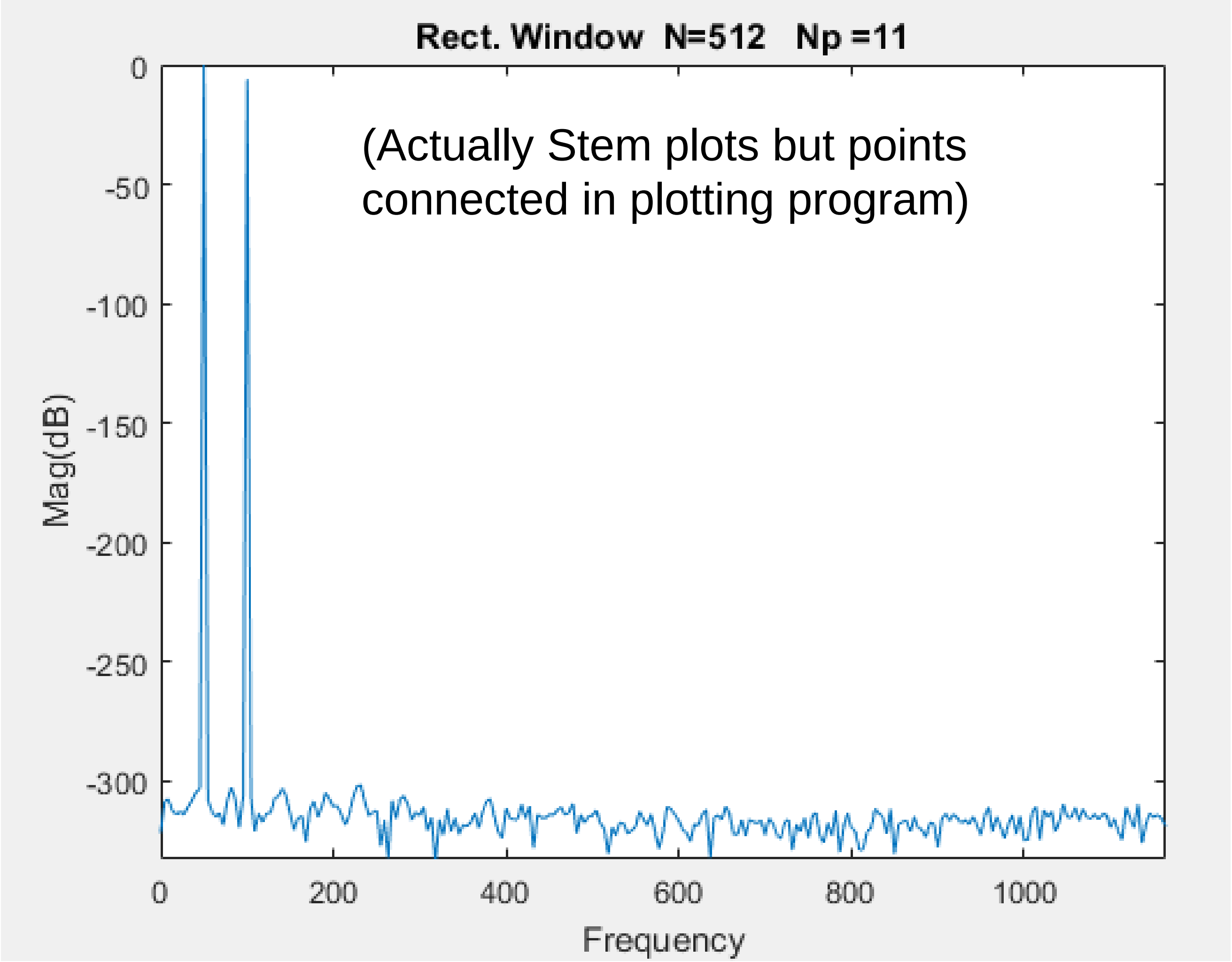
Spectral Response (expressed in dB)



Note Magnitude is Symmetric wrt $f_{SAMPLE}/2$

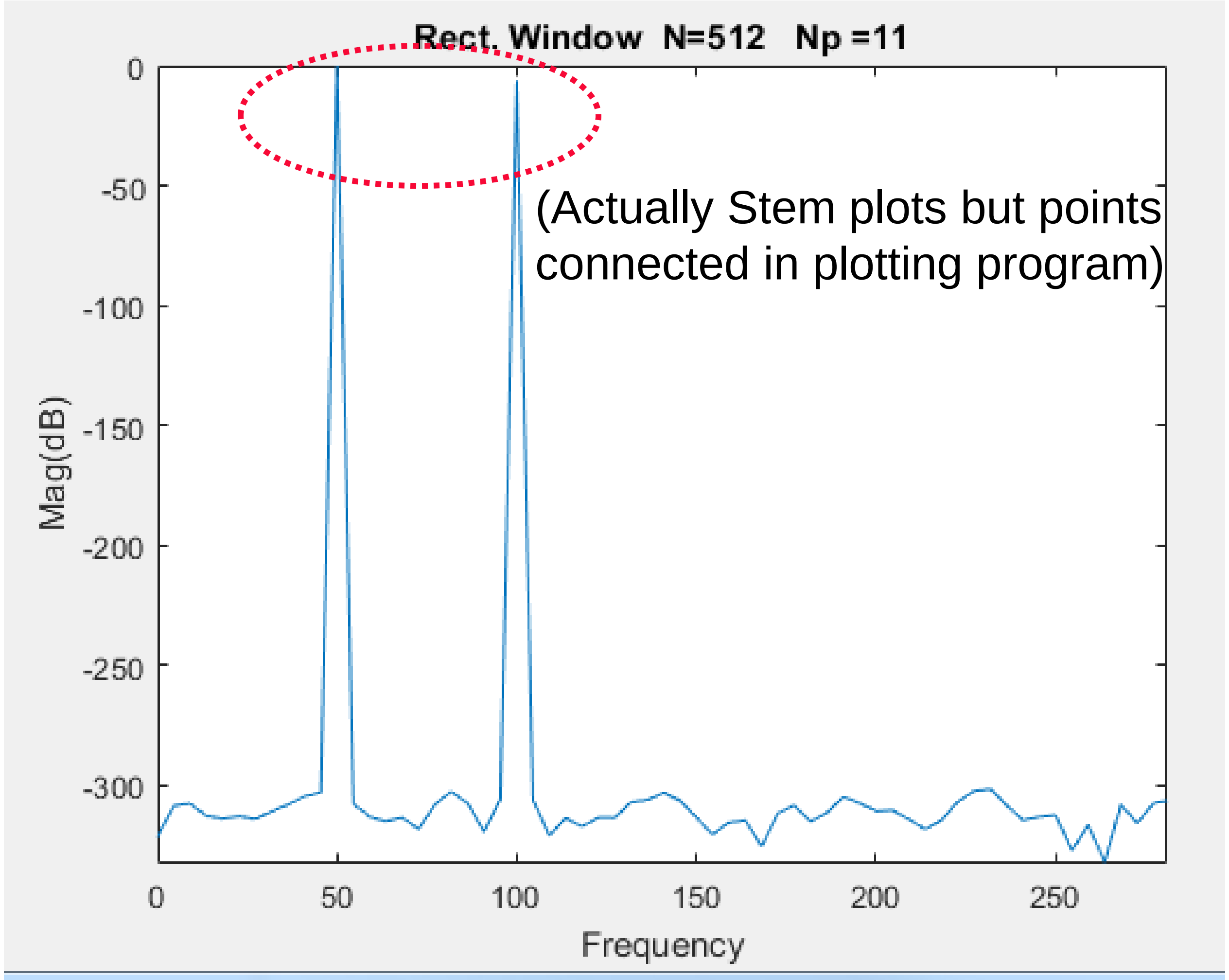
$$f_{AXIS} = f_{SIGNAL} \frac{k-1}{N_P}$$

Spectral Response (expressed in dB)



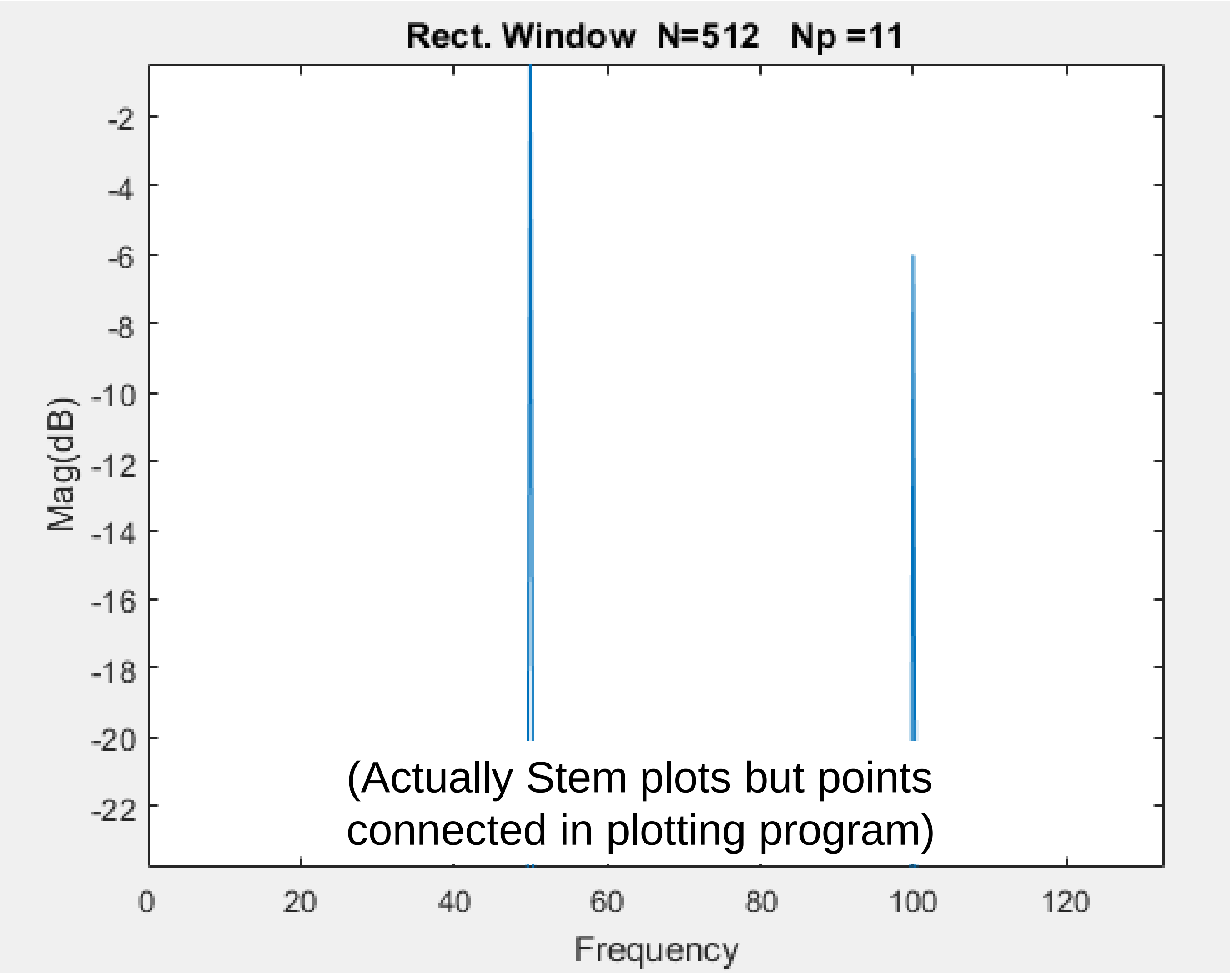
$$f_{AXIS} = f_{SIGNAL} \frac{k-1}{N_P}$$

Spectral Response



$$f_{AXIS} = f_{SIGNAL} \frac{k-1}{N_P}$$

Spectral Response



$$f_{AXIS} = f_{SIGNAL} \frac{k-1}{N_P}$$

Fundamental will appear at position 1+Np = 12

Columns 1 through 11

-321.7231 -308.6975 -307.7331 -312.9228 -314.0436 -313.1052 -314.1937 -311.0721 -308.1500 -304.8602 -303.1474

Columns 12 through 22

-0.0000 -307.8133 -313.2869 -315.1953 -313.5456 -318.5818 -308.0412 -302.7892 -307.6748 -319.5537 -306.1232

Columns 23 through 33

-6.0206 -306.3159 -321.1897 -313.7555 -317.4130 -313.5698 -313.6783 -307.1848 -306.3061 -303.1651 -306.5553

Columns 34 through 44

-313.5577 -320.6716 -315.7210 -314.8459 -325.6765 -311.7824 -308.5404 -315.3695 -311.5252 -304.9991 -307.6242

Columns 45 through 55

-310.8716 -310.5432 -314.2436 -318.5144 -314.7917 -307.0888 -302.5236 -301.7044 -308.4356 -314.7118 -313.2992

Observe system noise floor due to both spectral limitations of signal generator and numerical limitations in FFT are below -300db

Second Harmonic at 1+2Np = 23

Columns 1 through 11

-321.7231 -308.6975 -307.7331 -312.9228 -314.0436 -313.1052 -314.1937 -311.0721 -308.1500 -304.8602 -303.1474

Columns 12 through 22

-0.0000 -307.8133 -313.2869 -315.1953 -313.5456 -318.5818 -308.0412 -302.7892 -307.6748 -319.5537 -306.1232

Columns 23 through 33

-6.0206 -306.3159 -321.1897 -313.7555 -317.4130 -313.5698 -313.6783 -307.1848 -306.3061 -303.1651 -306.5553

Columns 34 through 44

-313.5577 -320.6716 -315.7210 -314.8459 -325.6765 -311.7824 -308.5404 -315.3695 -311.5252 -304.9991 -307.6242

Columns 45 through 55

-310.8716 -310.5432 -314.2436 -318.5144 -314.7917 -307.0888 -302.5236 -301.7044 -308.4356 -314.7118 -313.2992

Recall $20\log_{10}(0.5)=-6.0205999$

Third Harmonic at 1+3Np = 34

Columns 1 through 11

-321.7231 -308.6975 -307.7331 -312.9228 -314.0436 -313.1052 -314.1937 -311.0721 -308.1500 -304.8602 -303.1474

Columns 12 through 22

-0.0000 -307.8133 -313.2869 -315.1953 -313.5456 -318.5818 -308.0412 -302.7892 -307.6748 -319.5537 -306.1232

Columns 23 through 33

-6.0206 -306.3159 -321.1897 -313.7555 -317.4130 -313.5698 -313.6783 -307.1848 -306.3061 -303.1651 -306.5553

Columns 34 through 44

-313.5577 -320.6716 -315.7210 -314.8459 -325.6765 -311.7824 -308.5404 -315.3695 -311.5252 -304.9991 -307.6242

Columns 45 through 55

-310.8716 -310.5432 -314.2436 -318.5144 -314.7917 -307.0888 -302.5236 -301.7044 -308.4356 -314.7118 -313.2992

Example - Increasing N_P

WLOG assume f_{SIG}=50Hz

$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

Consider N_P=31 N=512

Example – Increasing N_P

WLOG assume f_{SIG}=50Hz

$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

$$f_{MAX\text{-}ACT} = 100\text{Hz}$$

Consider N_P=31 N=512

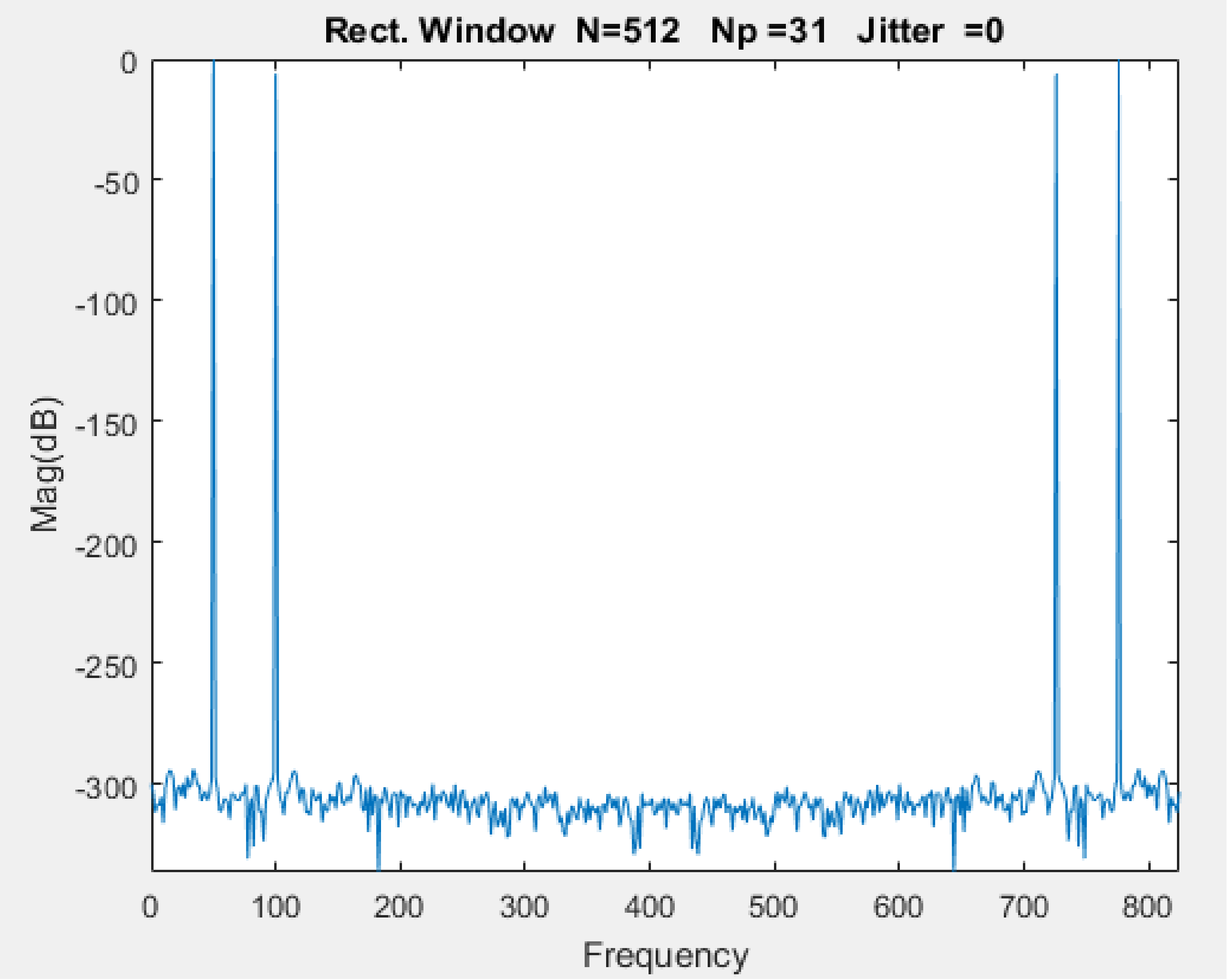
$$f_{MAX} = \frac{f_{SIG}}{2} \bullet \left[\frac{N}{N_P}\right] = \frac{50}{2} \bullet \frac{512}{31} = 412.9 \text{ Hz}$$

$$f_{MAX\text{-}ACT} << f_{MAX}$$

$$f_{SAMPLE} = \frac{1}{T_{SAMPLE}} = \frac{1}{\left(\dfrac{N_P \bullet T_{SIG}}{N}\right)} = \left[\frac{N}{N_P}\right] f_{SIG} = 2 f_{MAX} = 825.\tilde{8}\,Hz$$

Recall $20\log_{10}(1.0) = 0.0000000$

Recall $20\log_{10}(0.5) = -6.0205999$



Rect. Window N=512 Np =31 Jitter =0
0
-50
-100
-150
-200
-250
-300
Mag(dB)
0
100
200
300
400
500
600
700
800
Frequency

Fundamental will appear at position 1+Np = 32

Columns 1 through 11

-299.6472 -303.2647 -311.4997 -308.3073 -308.8025 -305.2792 -315.9958 -301.4260 -296.7862 -294.0674 -295.0438

Columns 12 through 22

-298.0298 -310.5850 -301.1461 -300.4606 -304.9693 -299.6351 -305.7190 -296.6982 -301.2474 -299.9385 -293.5499

Columns 23 through 33

-295.9396 -300.3529 -299.9751 -306.9248 -304.7769 -302.8556 -306.7452 -304.6290 -297.8691 -0.0000 -297.2993

Columns 34 through 44

-302.0105 -311.1106 -311.1480 -306.3087 -306.2084 -307.5956 -314.2196 -303.7547 -304.1900 -305.9837 -306.5840

Columns 45 through 55

-306.2397 -303.5875 -304.1769 -299.7907 -330.5678 -307.8635 -305.5380 -325.6389 -300.4851 -300.7637 -311.4764

Columns 56 through 66

-310.5216 -323.3310 -307.7034 -303.6633 -300.5140 -298.8582 -296.0678 -6.0206 -297.4520 -304.9769 -307.6024

Columns 67 through 77

-312.4863 -303.4440 -303.5171 -299.4001 -296.2720 -294.2365 -295.5554 -301.2603 -307.5154 -301.2971 -302.8147

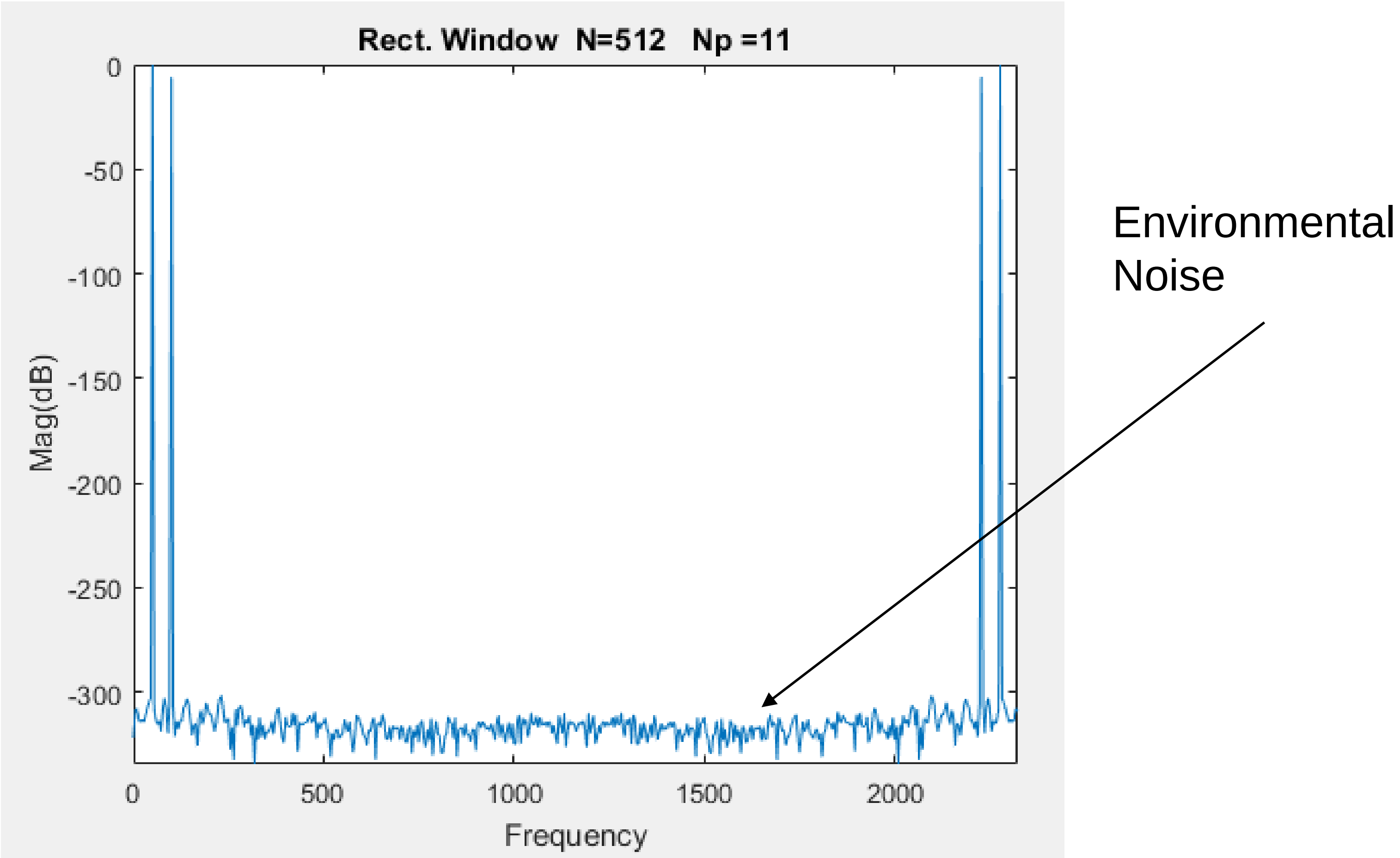
Columns 78 through 88

-311.7453 -310.8834 -312.7745 -301.5065 -304.5661 -306.9176 -305.4165 -303.5872 -315.6237 -308.0496 -310.6269

Columns 89 through 99

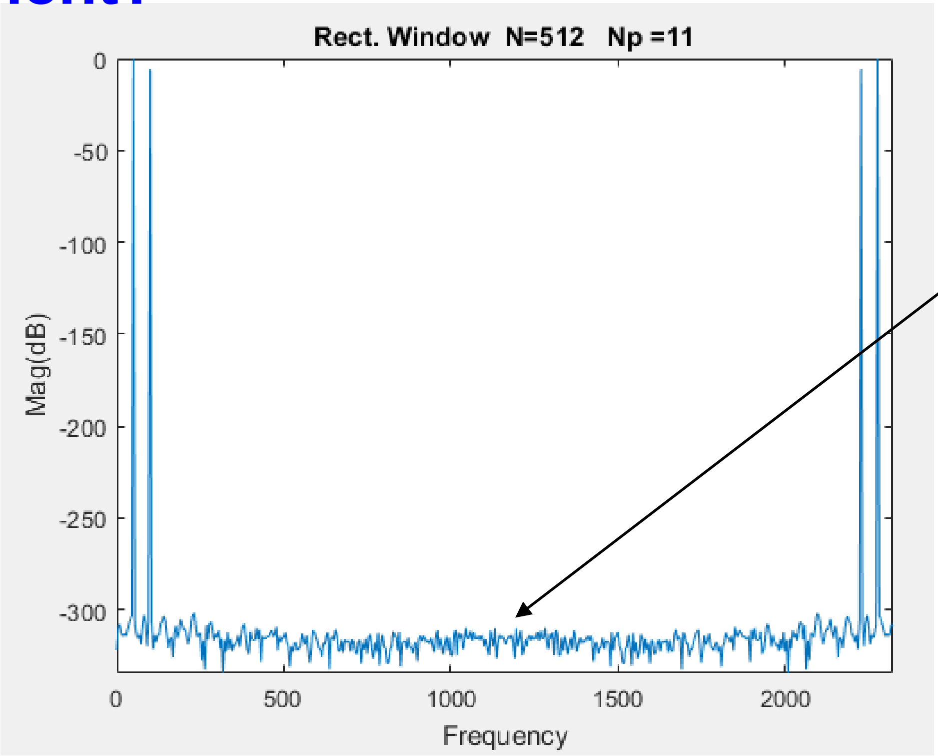
-311.1097 -303.8470 -307.2403 -304.9455 -310.2498 -299.7722 -298.9617 -307.3191 -308.4678 -306.2355 -304.7098

Question: How much noise is in the computational environment?



Is this due to quantization in the computational environment?
to numerical rounding in the FFT calculation?
to errors in calculating sin(x) ?

Question: How much noise is in the computational environment?



Environmental Noise

Observation: This noise is nearly uniformly distributed
The level of this noise at each component is around -310dB

Question: How much noise is in the computational environment?

Assume A_k = -310 dB for $0 \le k \le N$

$$A_{kdB} = 20\log10 A_k \quad \Rightarrow \quad A_k = 10^{\frac{A_{kDB}}{20}}$$

$$A_k \cong 10^{\frac{-310}{20}} = 10^{-15.5} \overset{defn}{=} \bar{A}$$

$$V_{Noise,RMS} \cong \sqrt{\sum_{k=1}^{N-1}\left(\frac{A_k}{\sqrt{2}}\right)^2} \overset{\substack{A_k=\bar{A} \\ N\ large}}{=} \bar{A}\sqrt{\frac{N}{2}}$$

$$V_{Noise,RMS} \cong \bar{A}\sqrt{\frac{N}{2}} = 10^{-15.5}\sqrt{\frac{512}{2}} = 5.1 \bullet 10^{-15} \cong 5fV$$

Note: **This computational environment has a very low total computational noise and does not become significant until the 46-bit resolution level is reached !!**

Tool Validation (MATLAB) ?

Likely does not cause significant errors for existing data converter spectral characterization applications

Likely can't attribute unexpected results in a design to MATLAB limitations for spectral characterization

Considerations for Spectral Characterization

- Tool Validation
- FFT Length
- Importance of Satisfying Hypothesis
- Windowing

Example – Increasing N

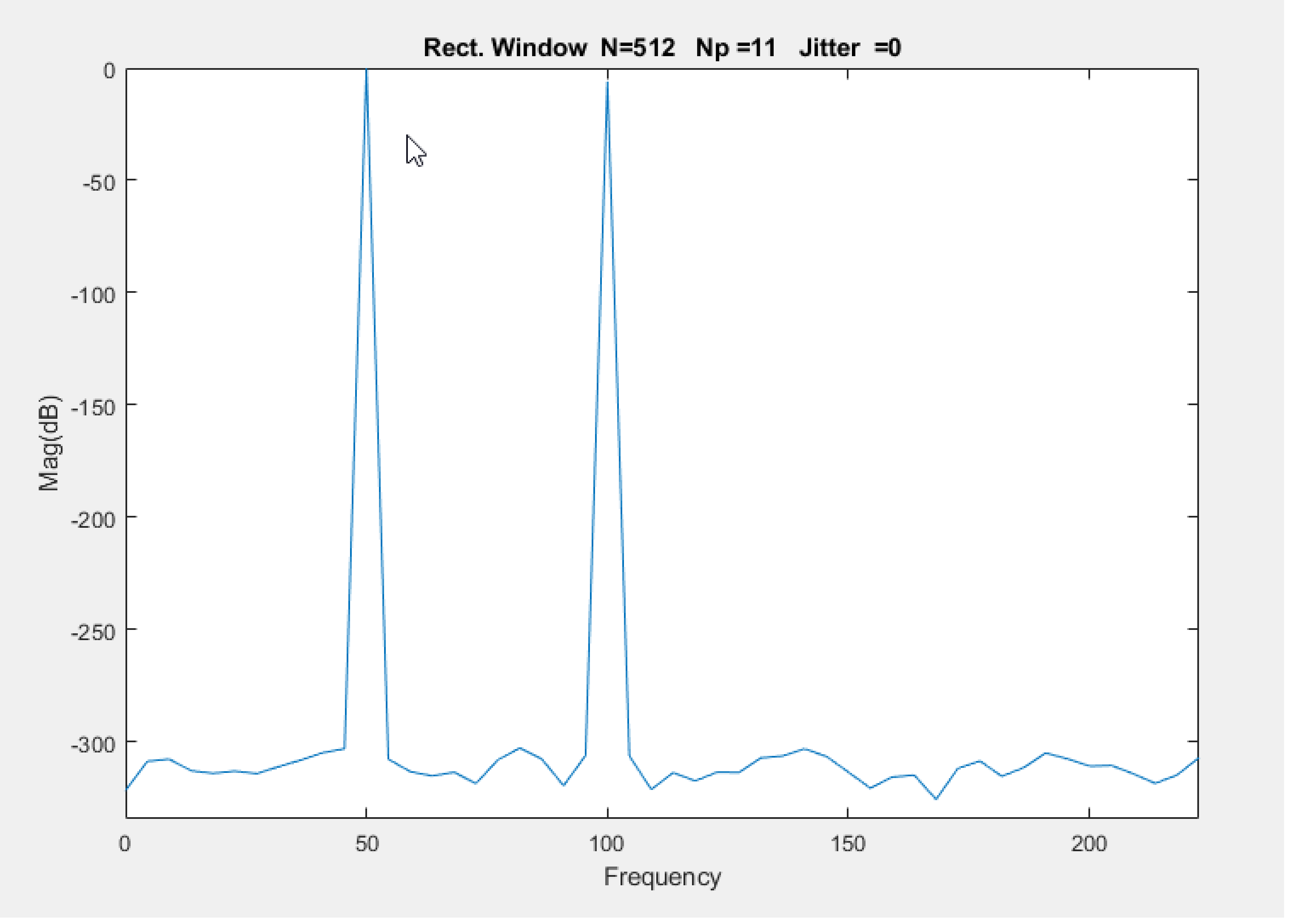
WLOG assume f_{SIG}=50Hz

$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

Recall N_P=11 N=512

Spectral Response



DFT Horizontal Axis Converter to Frequency : $f_{AXIS} = f_{SIGNAL} \frac{n-1}{N_P}$

Fundamental will appear at position 1+Np = 12
Second harmonic will appear at position $1+2N_p=23$

Columns 1 through 5

-321.7231 -308.6975 -307.7331 -312.9228 -314.0436

Columns 6 through 10

-313.1052 -314.1937 -311.0721 -308.1500 -304.8602

Columns 11 through 15

-303.1474 -0.0000 -307.8133 -313.2869 -315.1953

Columns 16 through 20

-313.5456 -318.5818 -308.0412 -302.7892 -307.6748

Columns 21 through 25

-319.5537 -306.1232 -6.0206 -306.3159 -321.1897

Recall system noise floor due to both spectral limitations of signal generator and numerical limitations in FFT are below -300db

Example – Increasing N

WLOG assume f_{SIG}=50Hz

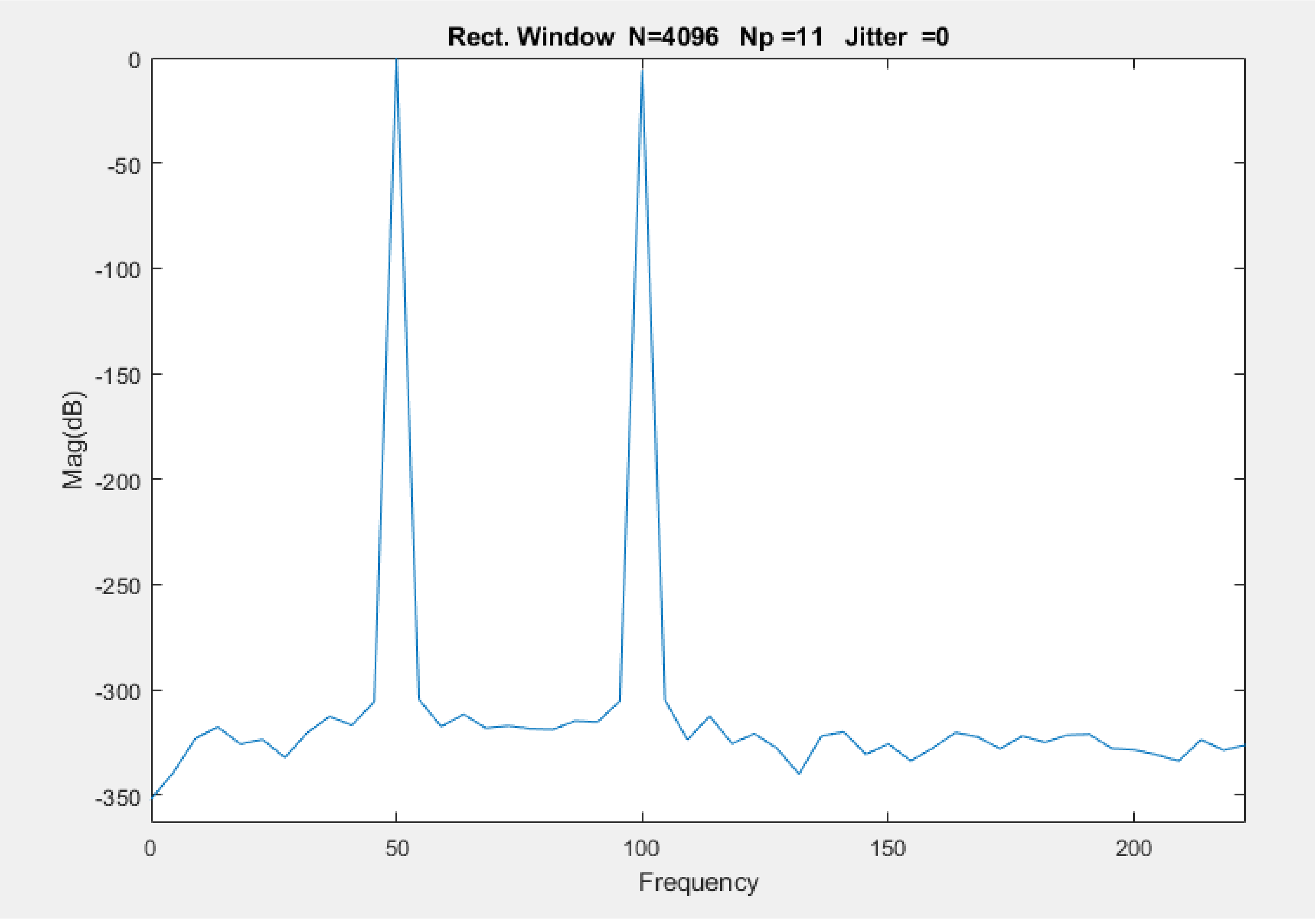
$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

- Increase length from 512 to 4096

Consider N_P=11 N=4096

Spectral Response



Fundamental will appear at position 1+Np = 12

Second harmonic will appear at position 1+2Np = 23

For Np=11

Columns 1 through 5

-351.9112 -339.3553 -322.9718 -317.6116 -325.7202

Columns 6 through 11

-323.6033 -332.1468 -320.3687 -312.6237 -316.7584

Columns 12 through 15

-305.5856 0 -304.4897 -317.4074 -311.6053

Columns 16 through 20

-318.1774 -317.0160 -318.4626 -318.7780 -314.7295

Columns 21 through 25

-315.2757 -305.2332 -6.0206 -304.8544 -323.7100

Third harmonic will appear at position 1+3Np = 34

For Np=11

Columns 26 through 30

-312.4569 -325.5267 -320.8299 -327.6482 -340.0002

Columns 31 through 35

-321.9740 -319.8910 -330.5761 -325.5995 -333.6951

Example - Increasing N_P

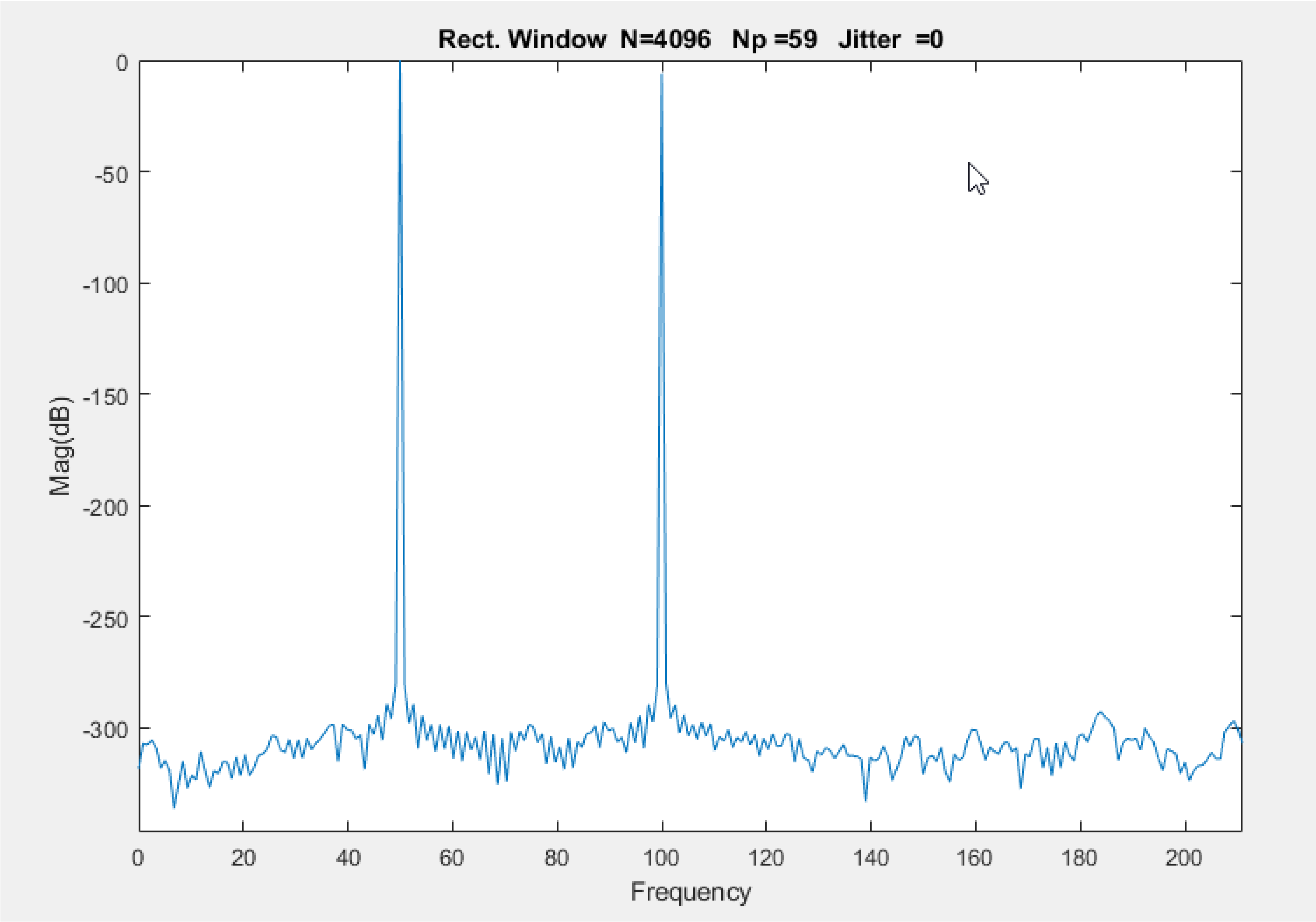
WLOG assume f_{SIG}=50Hz

$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

Consider N_P=59 N=4096

Spectral Response



Fundamental will appear at position 1+Np = 60

N_P=59 N=4096

Columns 1 through 13

-318.5027 -307.1222 -307.5852 -305.4679 -309.0657 -318.1047 -314.6599 -319.2843 -336.2985 -325.1317 -315.0935 -327.1326 -321.3676

Columns 14 through 26

-323.4308 -310.6280 -319.2470 -326.8257 -319.4502 -320.5664 -315.1544 -315.5020 -322.8177 -313.0289 -321.4672 -311.9602 -321.4265

Columns 27 through 39

-318.1381 -312.5542 -311.6208 -309.5333 -303.4393 -304.0017 -309.8193 -311.0570 -305.3422 -313.8205 -305.4006 -313.4782 -304.6525

Columns 40 through 52

-309.6946 -306.9554 -304.8106 -302.1759 -299.0381 -298.4703 -315.1285 -298.2425 -300.8413 -301.0957 -305.0282 -303.3066 -318.7951

Columns 53 through 65

-298.2648 -302.9334 -294.3147 -305.4528 -289.2434 -295.8942 -280.1798 -0.0000 -280.5964 -297.8514 -289.3267 -309.3347 -294.5964

Second Harmonic will appear at position 1+2Np = 119

N_P=59 N=4096

Columns 92 through 104

-302.8127 -316.2949 -303.7744 -315.5271 -308.4771 -318.7071 -304.6313 -318.0416 -306.1670 -308.5714 -302.7302 -302.2368 -299.1027

Columns 105 through 117

-308.9464 -297.3257 -301.3703 -300.2894 -306.4180 -304.1136 -311.0688 -297.7426 -306.8213 -294.6257 -309.2568 -289.3121 -297.4441

Columns 118 through 130

-280.6899 -6.0206 -280.0699 -295.8322 -289.6408 -302.2283 -294.1945 -303.4744 -298.5343 -305.2095 -297.5745 -303.8141 -298.1884

Columns 131 through 143

-310.0974 -303.9414 -305.9561 -300.8293 -308.8257 -304.3097 -306.3100 -301.7031 -307.6105 -303.0237 -312.6641 -304.8315 -309.8784

Third Harmonic will appear at position 1+3Np = 178

N_P=59 N=4096

Columns 157 through 169

-310.2143 -313.7396 -310.6594 -307.4932 -312.5744 -312.4317 -312.9279 -313.8457
-333.2123 -313.2139 -314.8757 -313.6770 -308.2654

Columns 170 through 182

-312.9108 -323.6234 -318.4068 -313.1440 -303.9230 -308.4125 -303.5257 -304.4918
-320.8338 -313.5467 -312.3853 -315.2628 -308.7278

Columns 183 through 195

-319.9543 -324.3752 -311.8755 -314.5450 -313.2239 -305.2555 -300.7224 -301.1777
-307.2956 -314.6381 -308.5318 -310.5178 -311.8403

Columns 196 through 208

-307.0945 -306.2901 -310.7842 -309.0464 -327.4021 -311.5712 -312.9993 -305.0627
-304.9283 -317.9667 -308.8503 -321.4805 -306.9463

Fundamental will appear at position 1+Np = 60

N_P=59 N=4096

Has the environmental noise floor increased?

Columns 1 through 13

-318.5027 -307.1222 -307.5852 -305.4679 -309.0657 -318.1047 -314.6599 -319.2843 -336.2985 -325.1317 -315.0935 -327.1326 -321.3676

Columns 14 through 26

-323.4308 -310.6280 -319.2470 -326.8257 -319.4502 -320.5664 -315.1544 -315.5020 -322.8177 -313.0289 -321.4672 -311.9602 -321.4265

Columns 27 through 39

-318.1381 -312.5542 -311.6208 -309.5333 -303.4393 -304.0017 -309.8193 -311.0570 -305.3422 -313.8205 -305.4006 -313.4782 -304.6525

Columns 40 through 52

-309.6946 -306.9554 -304.8106 -302.1759 -299.0381 -298.4703 -315.1285 -298.2425 -300.8413 -301.0957 -305.0282 -303.3066 -318.7951

Columns 53 through 65

-298.2648 -302.9334 -294.3147 -305.4528 -289.2434 -295.8942 -280.1798 -0.0000 -280.5964 -297.8514 -289.3267 -309.3347 -294.5964

N_P=59 N=4096

Second Harmonic will appear at position 1+2Np = 119

Has the environmental noise floor increased?

Columns 92 through 104

-302.8127 -316.2949 -303.7744 -315.5271 -308.4771 -318.7071 -304.6313 -318.0416 -306.1670 -308.5714 -302.7302 -302.2368 -299.1027

Columns 105 through 117

-308.9464 -297.3257 -301.3703 -300.2894 -306.4180 -304.1136 -311.0688 -297.7426 -306.8213 -294.6257 -309.2568 -289.3121 -297.4441

Columns 118 through 130

-280.6899 -6.0206 -280.0699 -295.8322 -289.6408 -302.2283 -294.1945 -303.4744 -298.5343 -305.2095 -297.5745 -303.8141 -298.1884

Columns 131 through 143

-310.0974 -303.9414 -305.9561 -300.8293 -308.8257 -304.3097 -306.3100 -301.7031 -307.6105 -303.0237 -312.6641 -304.8315 -309.8784

Has the environmental noise floor increased?

Example - Increasing N_P

WLOG assume f_{SIG}=50Hz

$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

Consider N_P=59 N=65536

Has the environmental noise floor increased?

Fundamental will appear at position 1+Np =60

N_P=59 N=65536

Columns 1 through 13

-334.8087 -306.4282 -305.3385 -306.0729 -310.4797 -319.1995 -319.2015 -326.9957 -323.5461 -320.9973 -322.3010 -330.1711 -341.5590

Columns 14 through 26

-335.6793 -324.5727 -331.9776 -328.7216 -328.2566 -322.5912 -323.1561 -327.1231 -324.6136 -327.6223 -319.4305 -319.3088 -320.0340

Columns 27 through 39

-333.4851 -317.9818 -317.6947 -310.0325 -305.2753 -304.6863 -313.1882 -316.6812 -308.8293 -320.0442 -315.7717 -321.1714 -314.0171

Columns 40 through 52

-311.9394 -313.5694 -312.8475 -313.0765 -311.4848 -309.8760 -311.2895 -317.5812 -311.1151 -305.7421 -310.4045 -307.3691 -308.2844

Columns 53 through 65

-303.0035 -311.0412 -301.9337 -305.0463 -295.1819 -297.5368 -286.7967 0 -287.9392 -297.7360 -295.6029 -303.5877 -302.3842

Has the environmental noise floor increased?

Second Harmonic will appear at position 1+2Np = 119 N_P=59 N=65536

Columns 92 through 104

-306.5163 -315.4662 -307.4725 -316.8662 -318.3799 -350.5195 -314.7483 -314.9668 -312.8332 -316.2735 -314.6227 -314.1683 -309.7421

Columns 105 through 117

-313.4242 -315.6570 -314.5305 -308.1299 -314.6932 -308.8232 -310.8042 -302.5240 -312.0286 -301.8591 -305.6318 -296.0575 -298.3169

Columns 118 through 130

-288.0902 -6.0206 -286.7600 -296.7222 -295.8080 -301.0121 -301.8241 -309.0274 -302.3173 -305.8535 -305.7722 -307.2773 -305.0839

Columns 131 through 143

-310.3001 -312.2301 -309.6370 -306.9460 -308.2486 -311.0093 -310.3350 -310.4601 -309.4687 -311.5006 -314.0138 -310.7739 -316.4801

Has the environmental noise floor increased?

Example - Increasing N_P

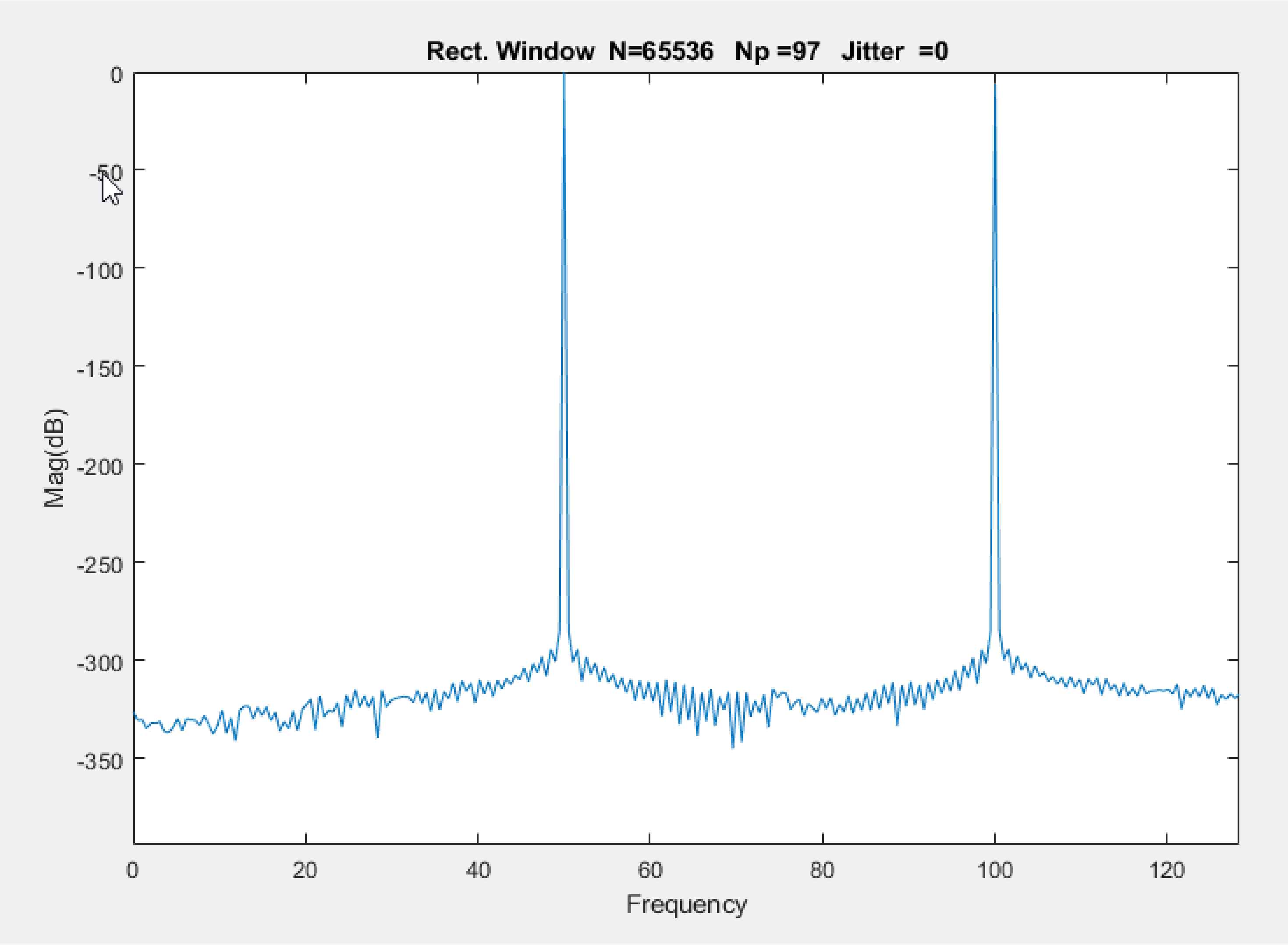
WLOG assume f_{SIG}=50Hz

$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

Consider N_P=97 N=65536

Spectral Response



Has the environmental noise floor increased?

Fundamental will appear at position 1+Np =98 N_P=97 N=65536

Columns 79 through 91

-309.9736 -317.2573 -310.8358 -318.6741 -310.3694 -314.6463 -309.3686 -312.1643
-307.6865 -309.9408 -303.7856 -310.9949 -301.7664

Columns 92 through 104

-306.4032 -298.2550 -308.0223 -294.4658 -300.6493 -284.9748 0 -284.9686
-300.8932 -294.4063 -310.6871 -298.2620 -306.9972

Columns 105 through 117

-301.4981 -310.9186 -303.6470 -311.3991 -307.0910 -314.2788 -309.0782 -316.6942
-309.7874 -320.7255 -310.2243 -320.0172 -310.9449

Has the environmental noise floor increased?

Second Harmonic will appear at position 1+2Np = 195 N_P=97 N=65536

Columns 170 through 182

-312.5506 -322.0807 -311.2396 -333.5575 -312.9314 -323.6003 -310.8598 -322.9989 -312.7526 -325.1455 -311.4532 -320.2566 -310.0410

Columns 183 through 195

-317.0955 -309.0581 -315.8184 -305.1452 -315.2504 -302.7042 -309.1048 -298.8906 -311.9708 -294.6866 -301.5044 -285.0441 -6.0206

Columns 196 through 208

-284.8891 -299.9611 -294.2706 -307.0905 -297.8530 -304.9661 -301.2032 -309.1250 -302.9453 -308.3046 -306.3036 -310.8076 -308.3658

Has the environmental noise floor increased?

Question: How much noise is in the computational environment?

Assume A_k = -310 dB for $0 \le k \le N$

$$A_{kdB} = 20\log10 A_k \quad \Rightarrow \quad A_k = 10^{\frac{A_{kDB}}{20}}$$

$$V_{Noise,RMS} \cong \bar{A}\sqrt{\frac{N}{2}} = 10^{-15.5}\sqrt{\frac{512}{2}} = 5.1 \bullet 10^{-15} \cong 5fV$$

N_P=31 N=512

Assume A_k = -310 dB for $0 \le k \le N$

N_P=97 N=65536

$$V_{Noise,RMS} \cong \bar{A}\sqrt{\frac{N}{2}} = 10^{-15.5}\sqrt{\frac{65536}{2}} = 57 \bullet 10^{-15} \cong 57fV$$

Note: **This computational environment is still very low even when N_p and N become large**

Considerations for Spectral Characterization

FFT Length

- FFT Length does not significantly affect the computational noise floor
- Although not shown here yet, FFT length does reduce the quantization noise floor coefficients

If we assume E_{QUANT} is fixed

$$E_{QUANT} \cong \sqrt{\sum_{k=2}^{2^{n_{DFT}}} A_k^2}$$

If the A_k's are constant and equal

$$E_{QUANT} \cong A_k 2^{n_{DFT}/2}$$

Solving for A_k, obtain

$$A_k \cong \frac{E_{QUANT}}{2^{n_{DFT}/2}}$$

If input is full-scale sinusoid with only amplitude quantization with n-bit res,

$$E_{QUANT} \cong \frac{X_{LSB}}{\sqrt{12}} = \frac{X_{REF}}{\sqrt{3} \bullet 2^{n+1}}$$

Considerations for Spectral Characterization

FFT Length

$$E_{QUANT} \cong \frac{X_{LSB}}{\sqrt{12}} = \frac{X_{REF}}{\sqrt{3} \bullet 2^{n+1}}$$

Substituting for E_{QUANT}, obtain

$$A_k \cong \frac{X_{REF}}{\sqrt{3} \bullet 2^{n+1} 2^{n_{DFT}/2}}$$

This value for A_k thus decreases with the length of the DFT window

Example: if n=16, n_{DFT}=12 (4096 pt transform), and X_{REF}=1V,
then A_k=6.9E-8V (-143dB),

(Note A_k>> computational noise for all practical n, n_{DFT})

Considerations for Spectral Characterization

- Tool Validation
- FFT Length
- Importance of Satisfying Hypothesis
  - NP is an integer
  - Band-limited excitation
- Windowing

Example

WLOG assume f_{SIG}=50Hz

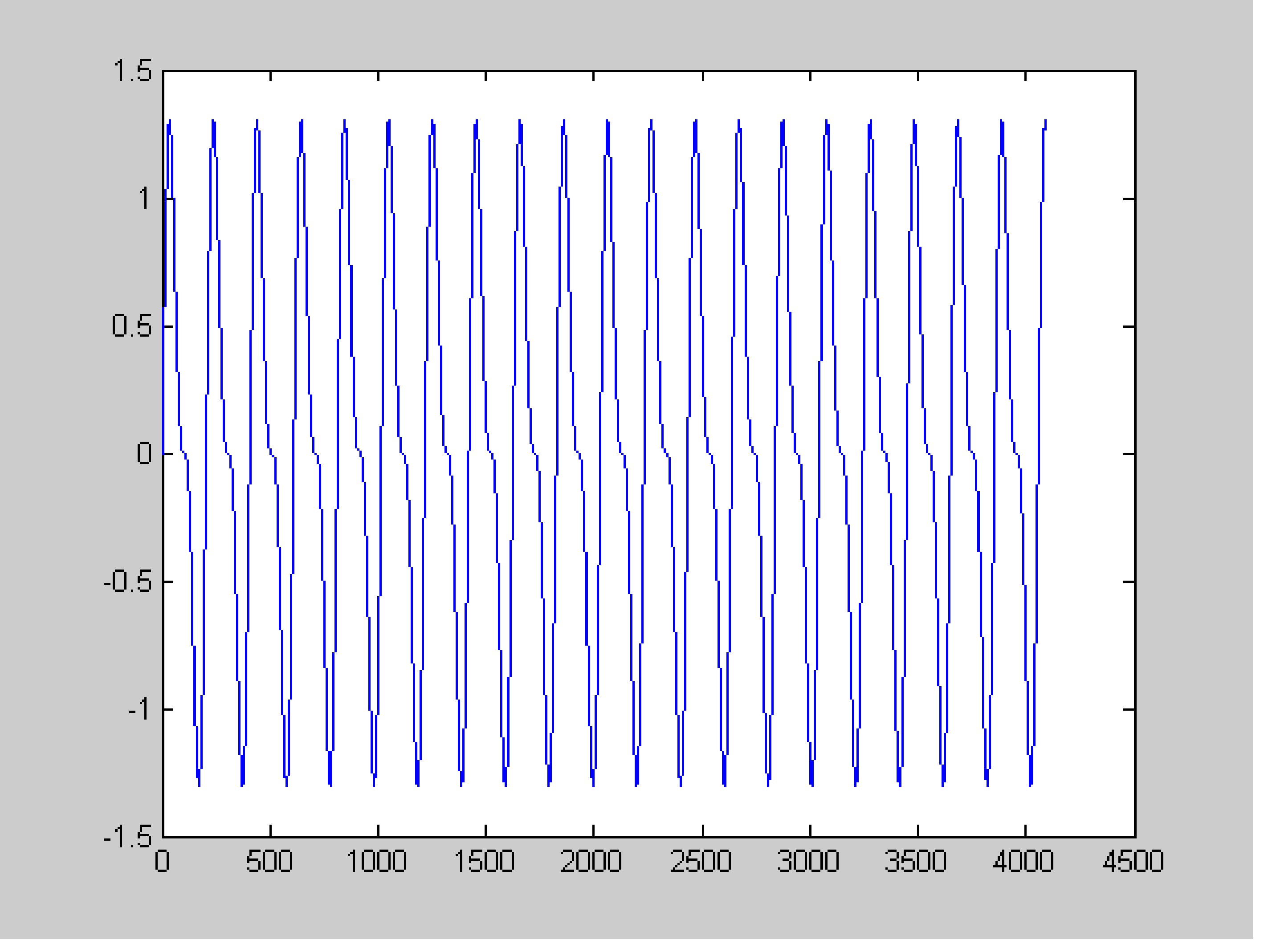
$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

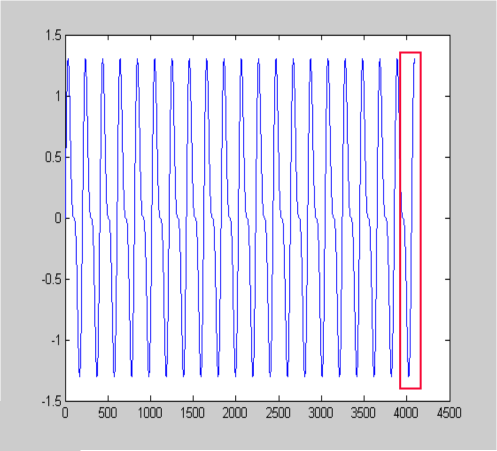
Consider N_P=20.2 N=4096

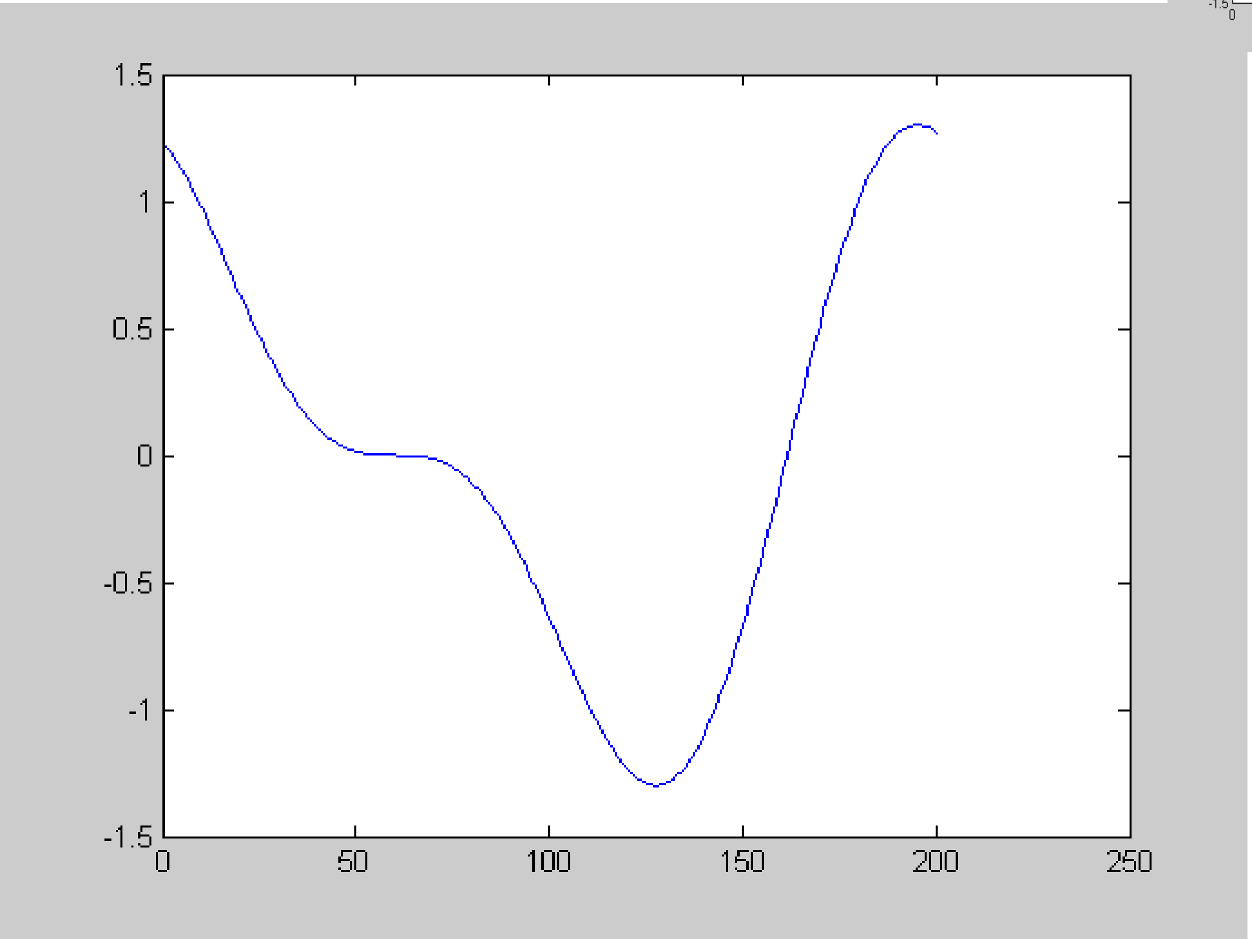
Recall $20\log_{10}(0.5)=-6.0205999$

Input Waveform

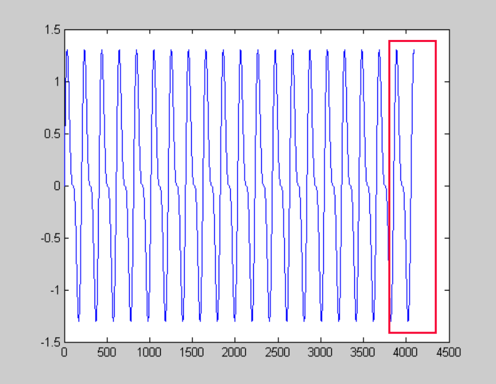


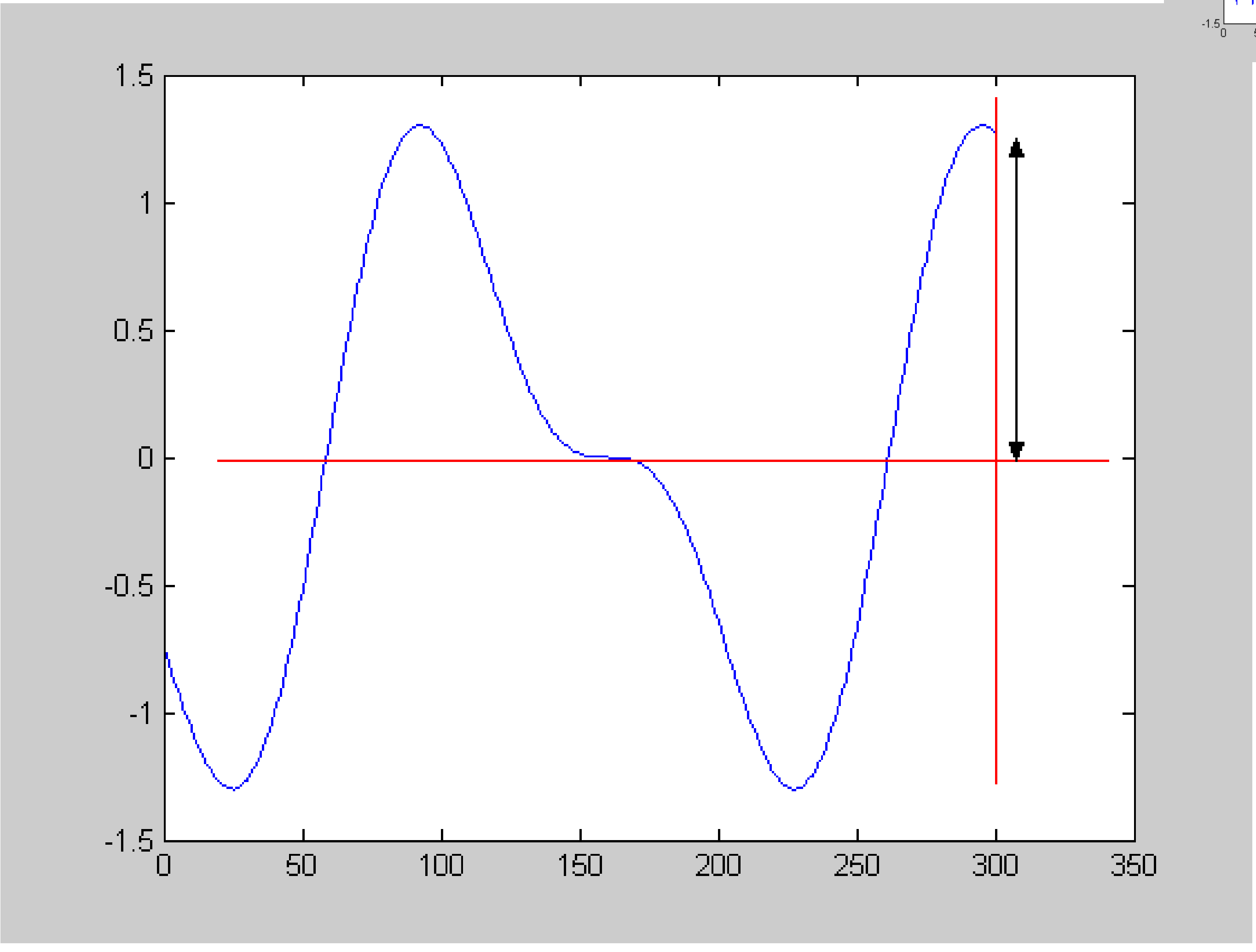
Input Waveform



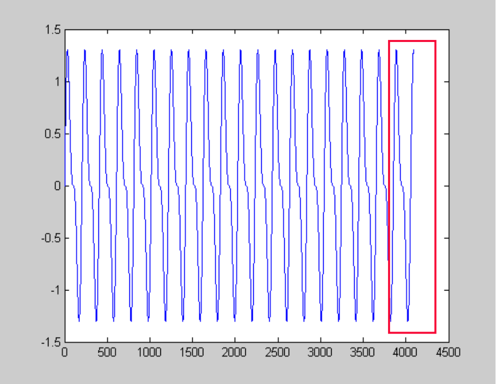


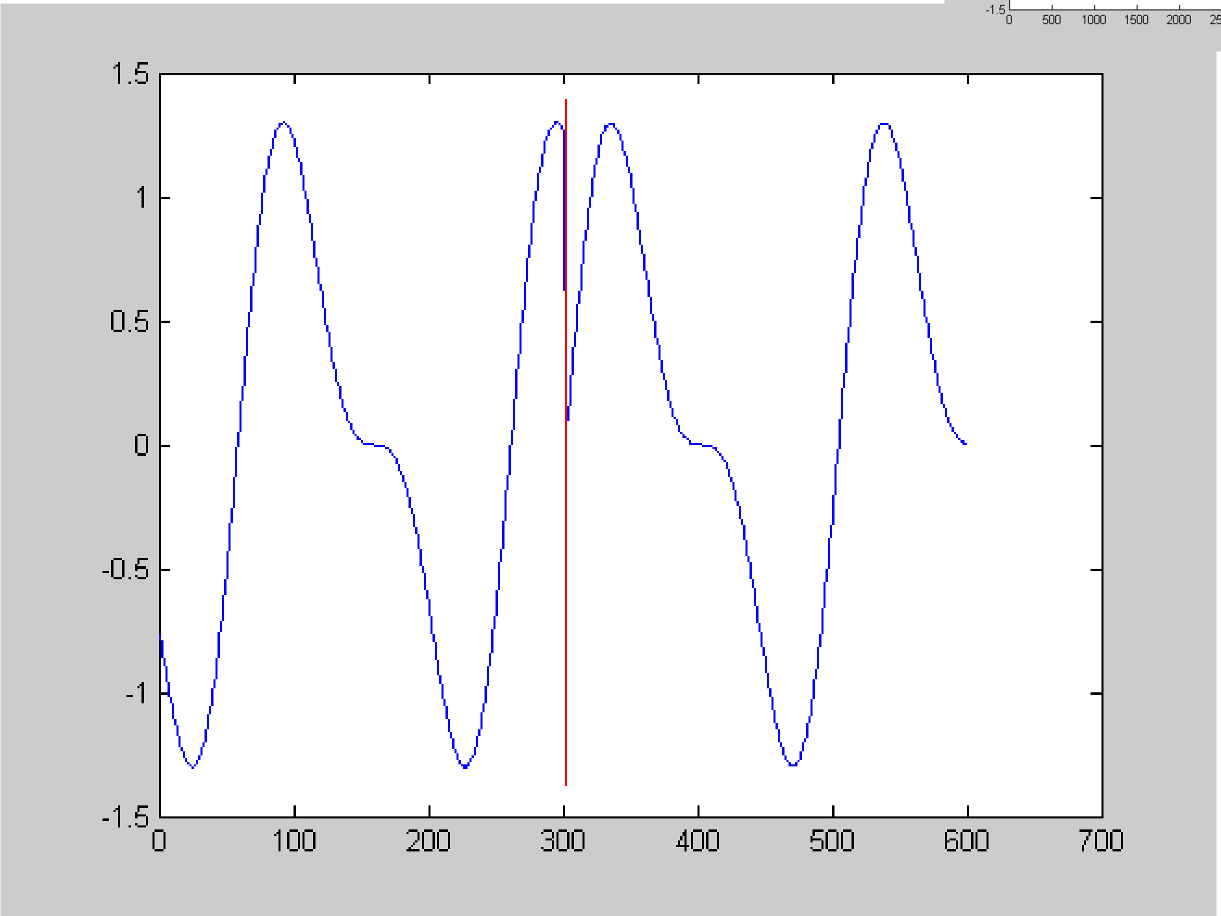
Input Waveform



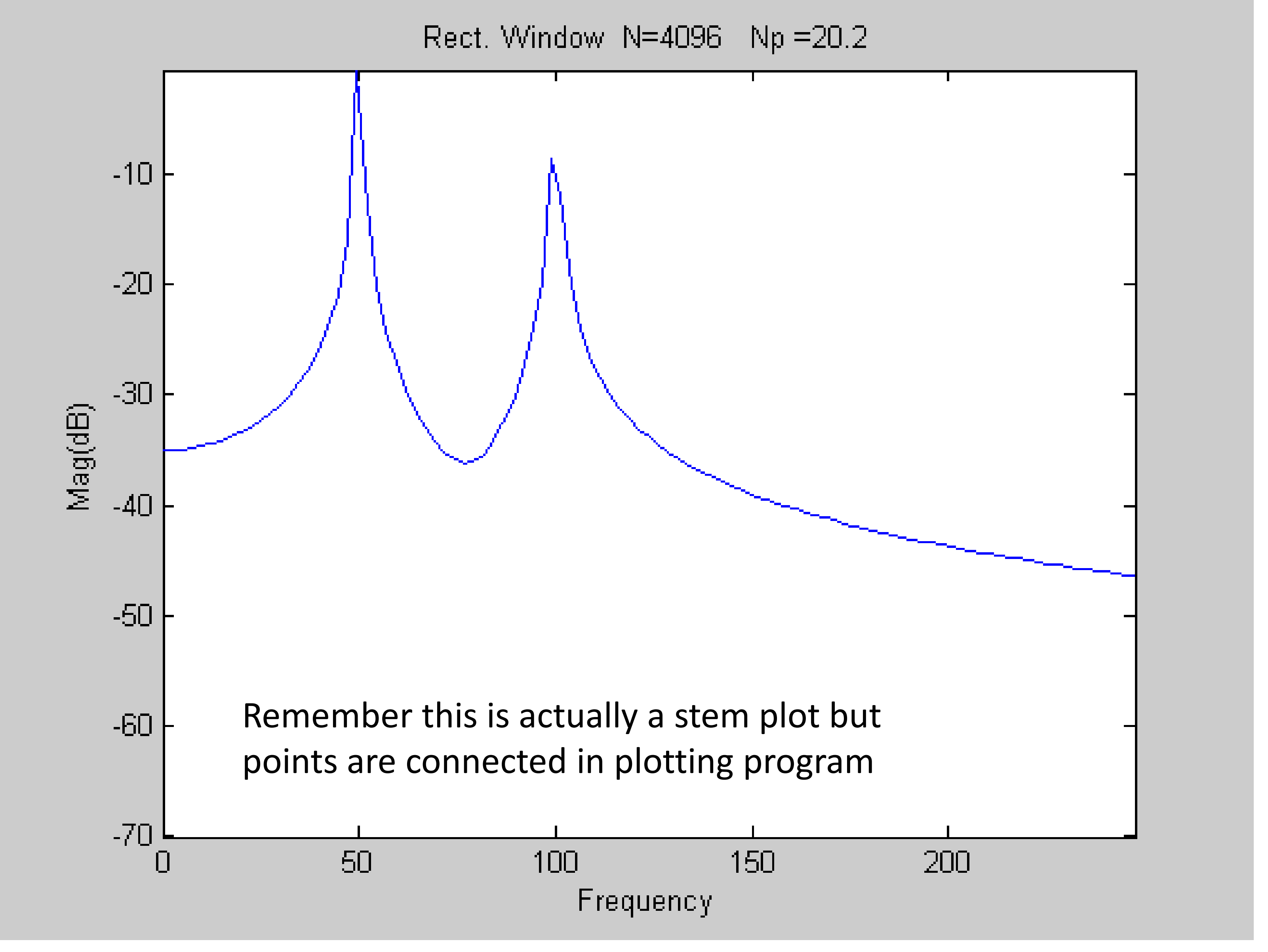


Input Waveform





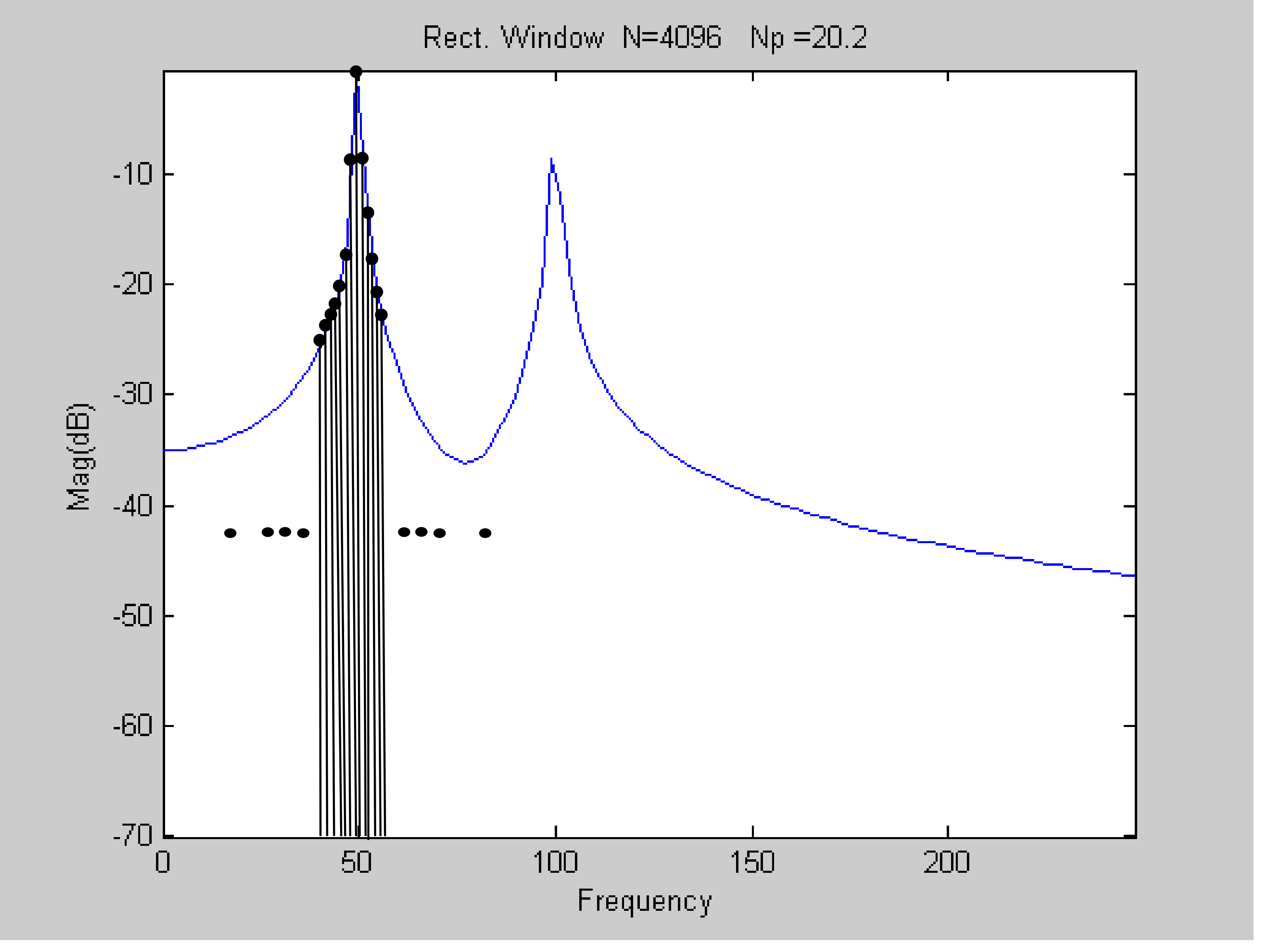
Spectral Response



Stay Safe and Stay Healthy !

End of Lecture 5

Spectral Response



Fundamental will appear at position 1+Np = 21

Columns 1 through 7

-35.0366 -35.0125 -34.9400 -34.8182 -34.6458 -34.4208 -34.1403

Columns 8 through 14

-33.8005 -33.3963 -32.9206 -32.3642 -31.7144 -30.9535 -30.0563

Columns 15 through 21

-28.9855 -27.6830 -26.0523 -23.9155 -20.8888 -15.8561 **-0.5309**

Columns 22 through 28

-12.8167 -20.1124 -24.2085 -27.1229 -29.4104 -31.2957 -32.8782

Columns 29 through 35

-34.1902 -35.2163 -35.9043 -36.1838 -35.9965 -35.3255 -34.1946

Note there is a dramatic increase in the noise floor and a significant change in and spreading of the fundamental!!

k^{th} harmonic will appear at position 1+k•Np

Columns 36 through 42

-32.6350 -30.6397 -28.1125 -24.7689 -19.7626 -8.5639 -11.7825

Columns 43 through 49

-20.0158 -23.9648 -26.5412 -28.4370 -29.9279 -31.1519 -32.1874

Columns 50 through 56

-33.0833 -33.8720 -34.5759 -35.2113 -35.7902 -36.3218 -36.8133

Columns 57 through 63

-37.2703 -37.6974 -38.0984 -38.4762 -38.8336 -39.1725 -39.4949

Columns 64 through 70

-39.8024 -40.0963 -40.3778 -40.6479 -40.9076 -41.1576 -41.3987

k^{th} harmonic will appear at position 1+k•Np

Columns 36 through 42

-32.6350 -30.6397 -28.1125 -24.7689 -19.7626 -8.5639 -11.7825

Columns 43 through 49

-20.0158 -23.9648 -26.5412 -28.4370 -29.9279 -31.1519 -32.1874

Columns 50 through 56

-33.0833 -33.8720 -34.5759 -35.2113 -35.7902 -36.3218 -36.8133

Columns 57 through 63

-37.2703 -37.6974 -38.0984 -38.4762 -38.8336 -39.1725 -39.4949

Columns 64 through 70

-39.8024 -40.0963 -40.3778 -40.6479 -40.9076 -41.1576 -41.3987

Observations

- Modest change in sampling window of 0.2 out of 20 periods (1%) results in a big error in both fundamental and harmonic
- More importantly, dramatic raise in the “noise floor” !!! (from over -300dB to only -12dB)

Example

WLOG assume f_{SIG}=50Hz

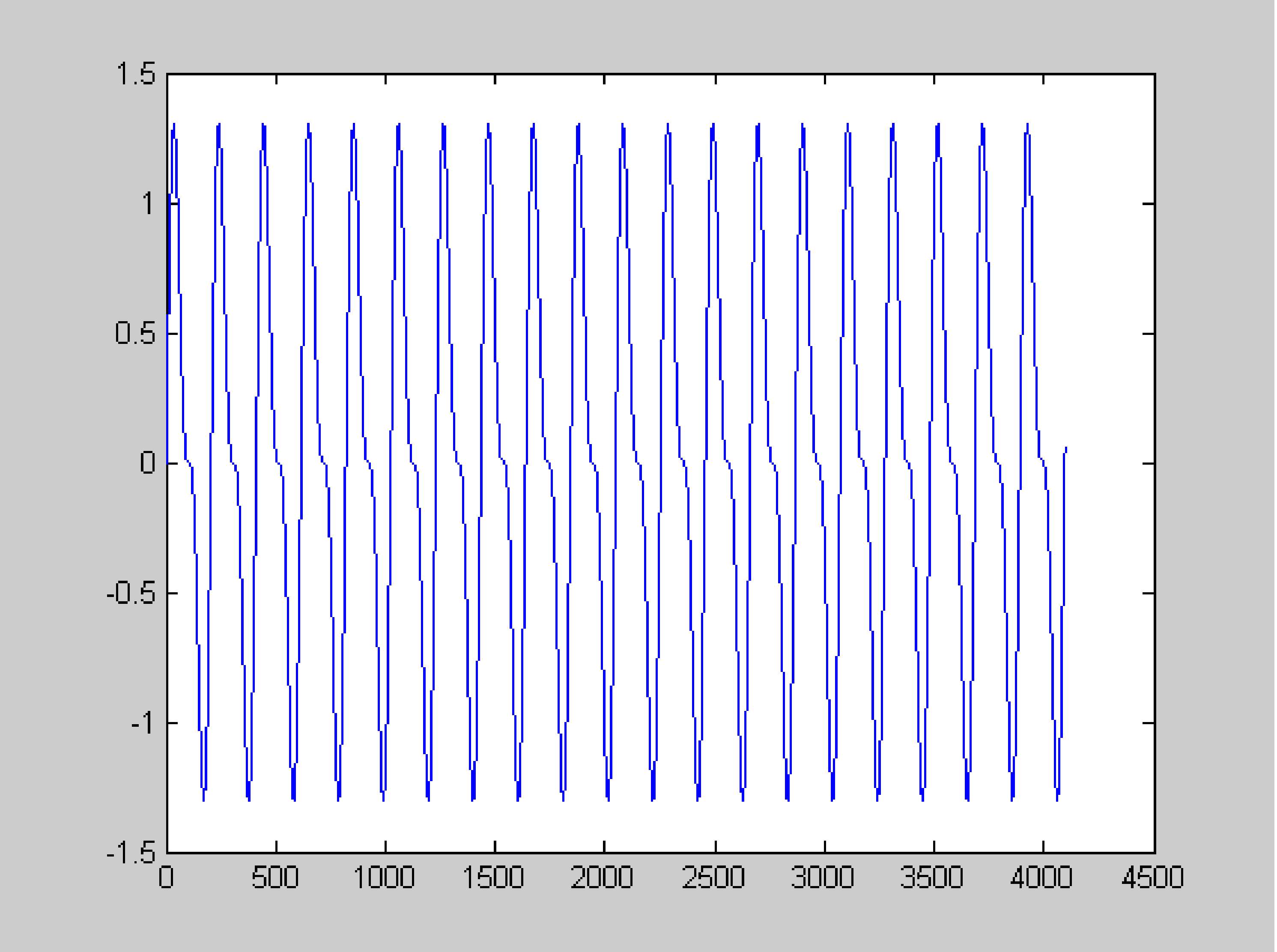
$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

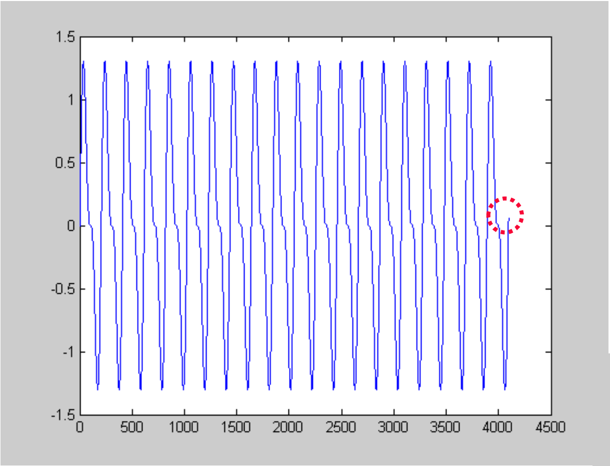
Consider N_P=20.01 N=4096

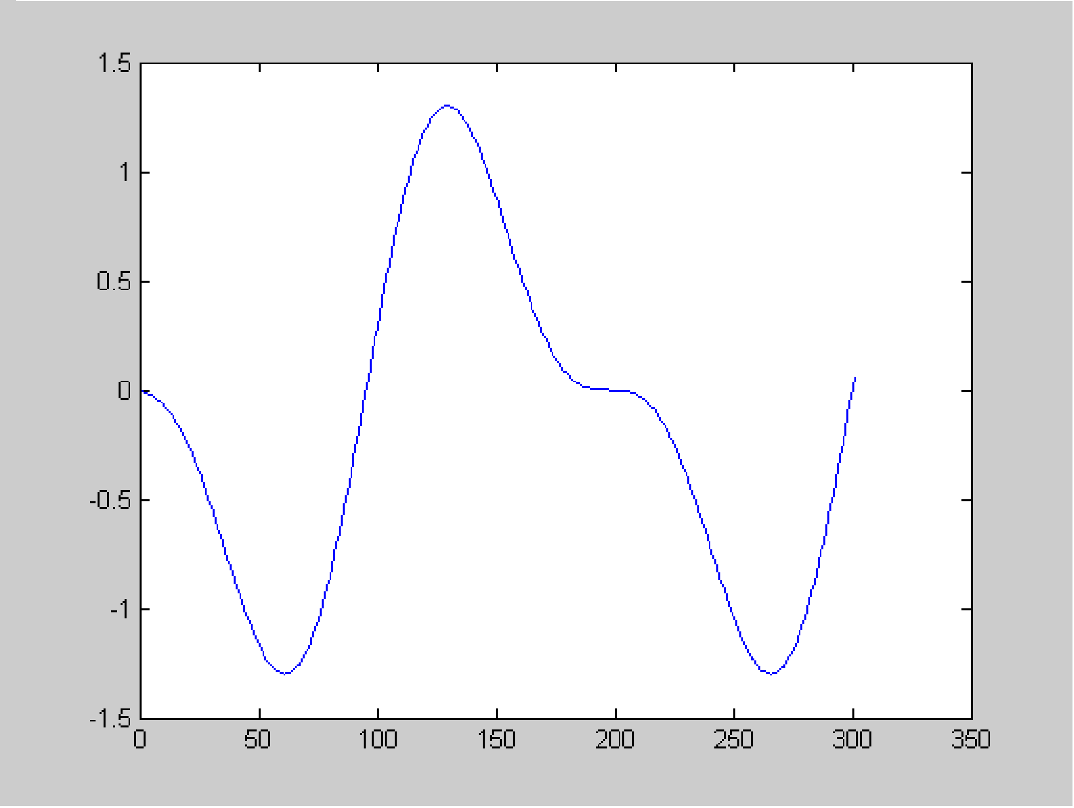
Deviation from hypothesis is .05% of the sampling window

Input Waveform

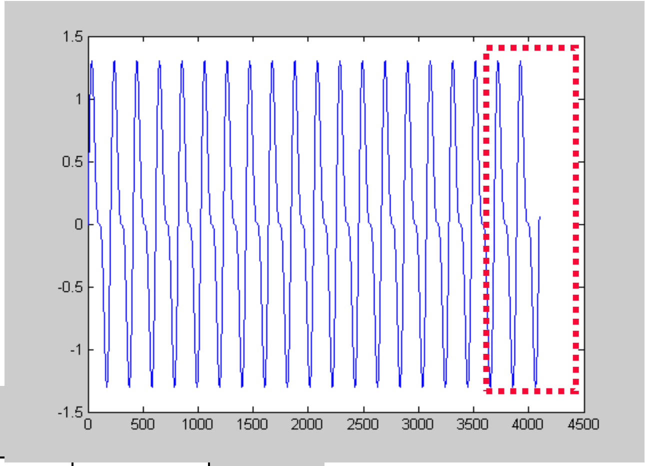


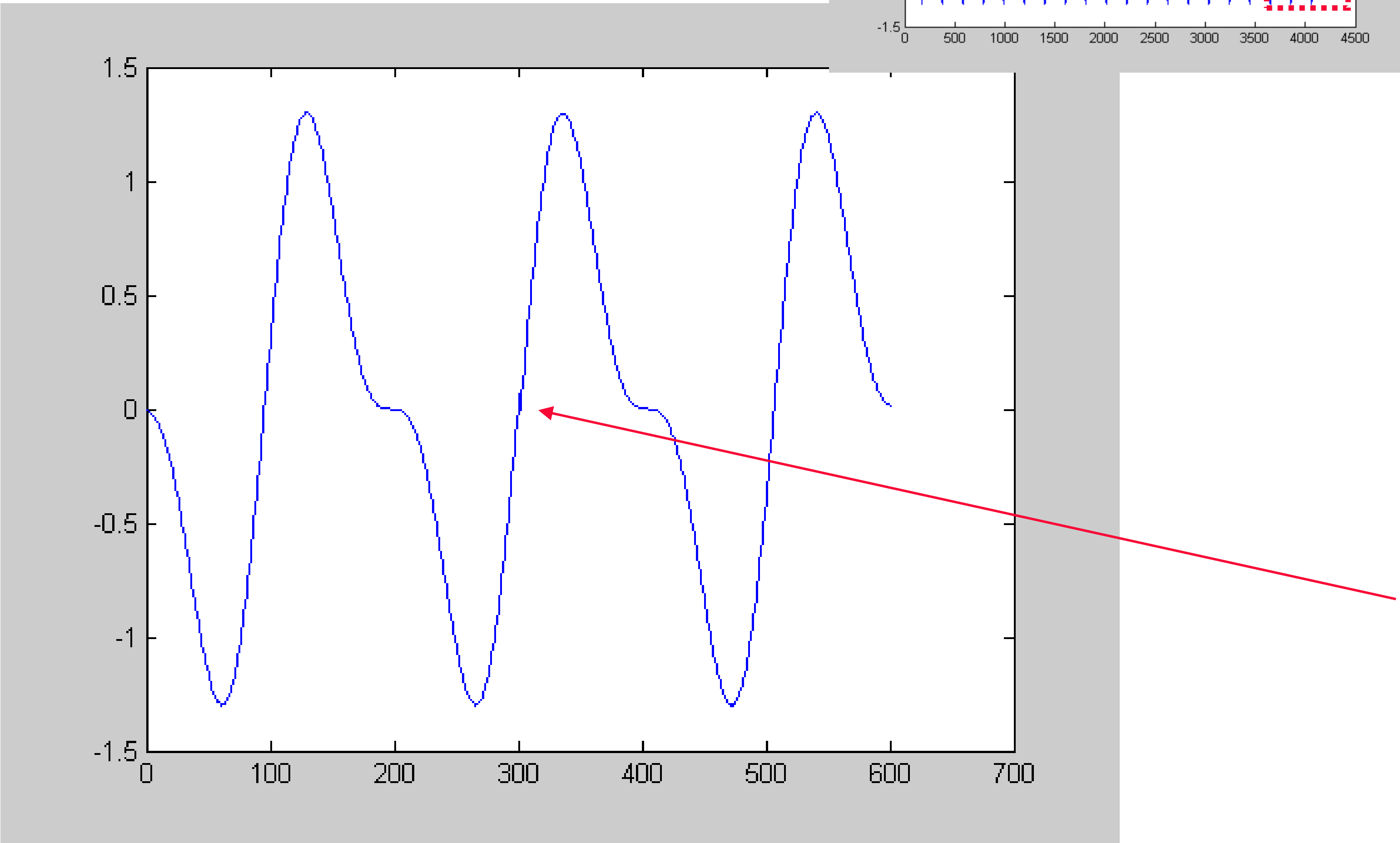
Input Waveform



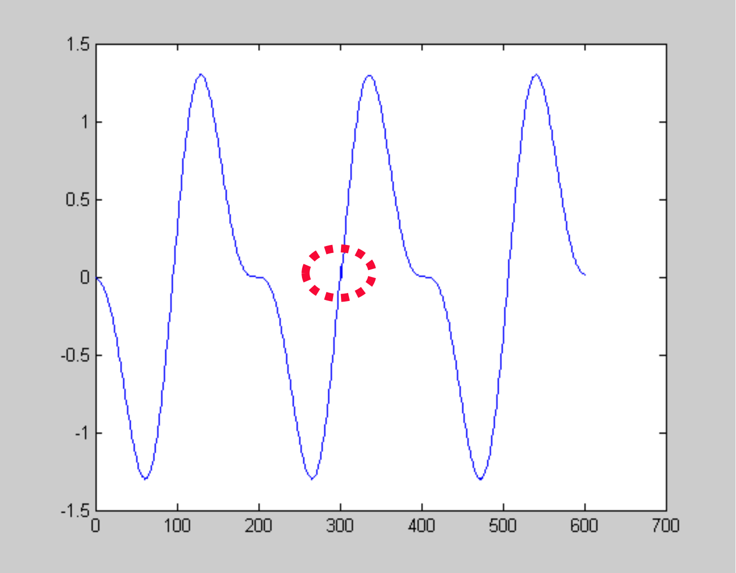


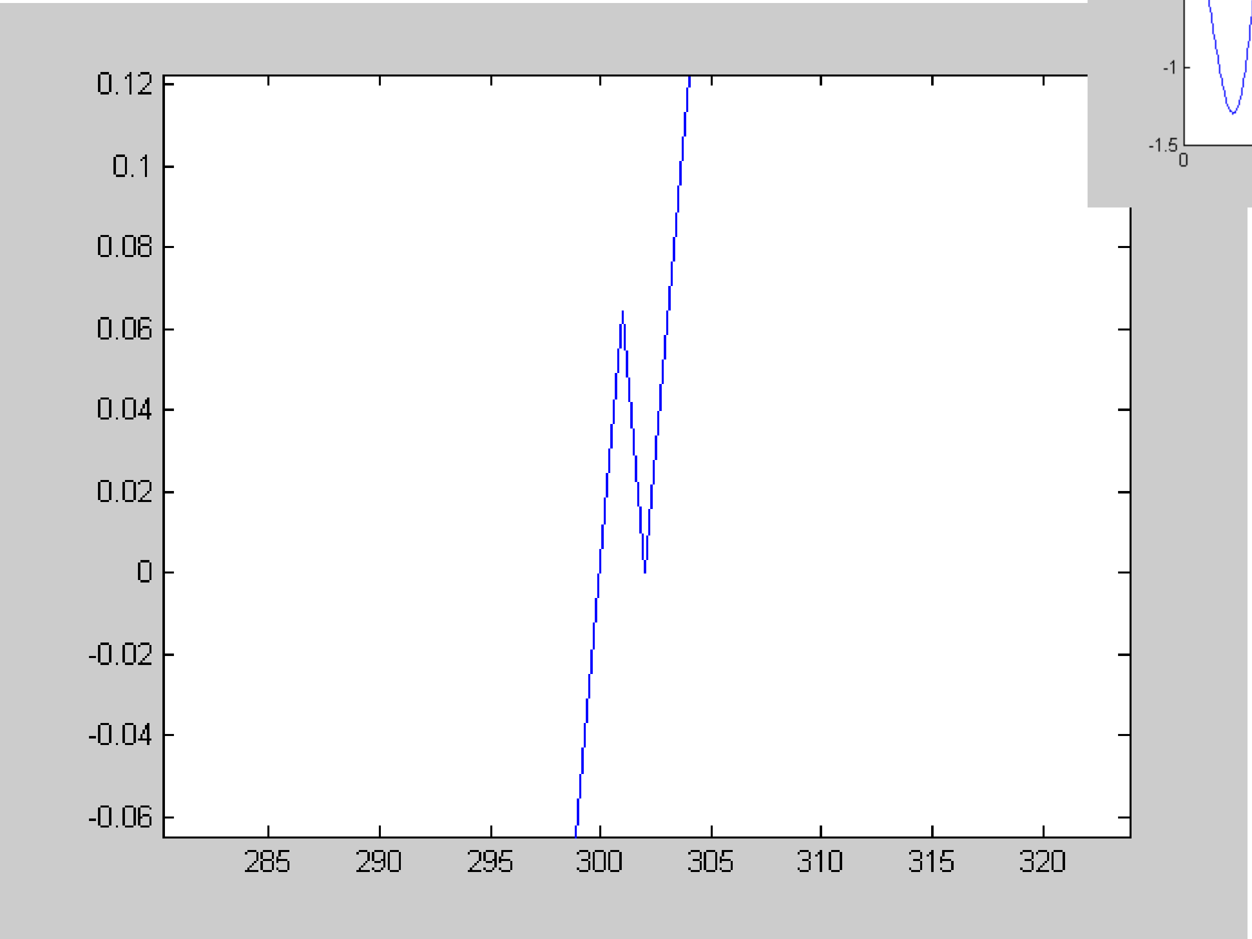
Input Waveform



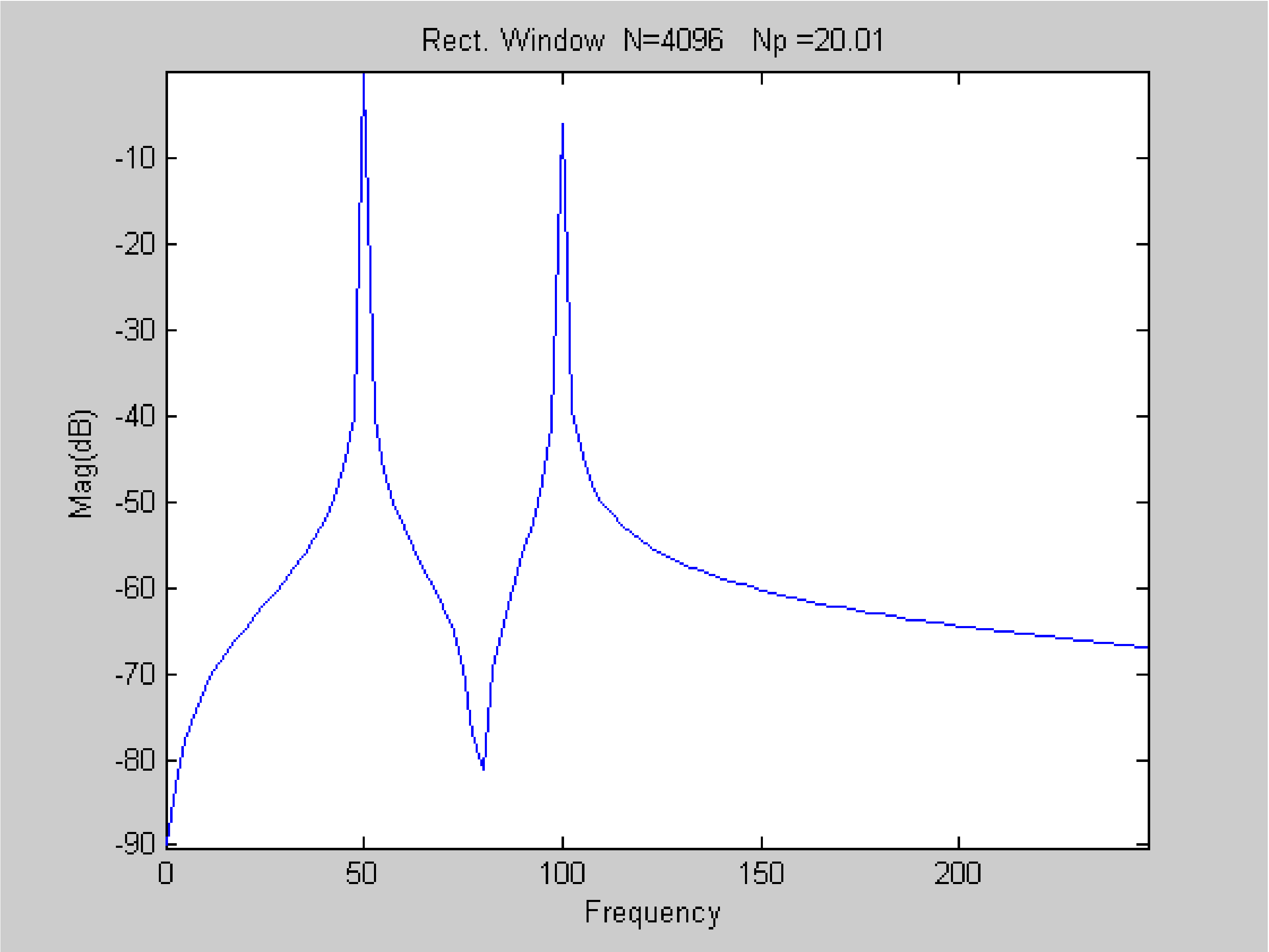


Input Waveform





Spectral Response with Non-Coherent Sampling



(zoomed in around fundamental)

Fundamental will appear at position 1+Np = 21

Columns 1 through 7

-89.8679 -83.0583 -77.7239 -74.2607 -71.6830 -69.5948 -67.8044

Columns 8 through 14

-66.2037 -64.7240 -63.3167 -61.9435 -60.5707 -59.1642 -57.6859

Columns 15 through 21

-56.0866 -54.2966 -52.2035 -49.6015 -46.0326 -40.0441 -0.0007

Columns 22 through 28

-40.0162 -46.2516 -50.0399 -52.8973 -55.3185 -57.5543 -59.7864

Columns 29 through 35

-62.2078 -65.1175 -69.1845 -76.9560 -81.1539 -69.6230 -64.0636

k^{th} harmonic will appear at position $1+k \cdot Np$

Columns 36 through 42

-59.9172 -56.1859 -52.3380 -47.7624 -40.9389 -6.0401 -39.2033

Observations

- Modest change in sampling window of 0.01 out of 20 periods (.05%) still results in a modest error in both fundamental and harmonic
- More importantly, substantial raise in the computational noise floor !!! (from over -300dB to only -40dB)
- Errors at about the 6-bit level !

Example

WLOG assume f_{SIG}=50Hz

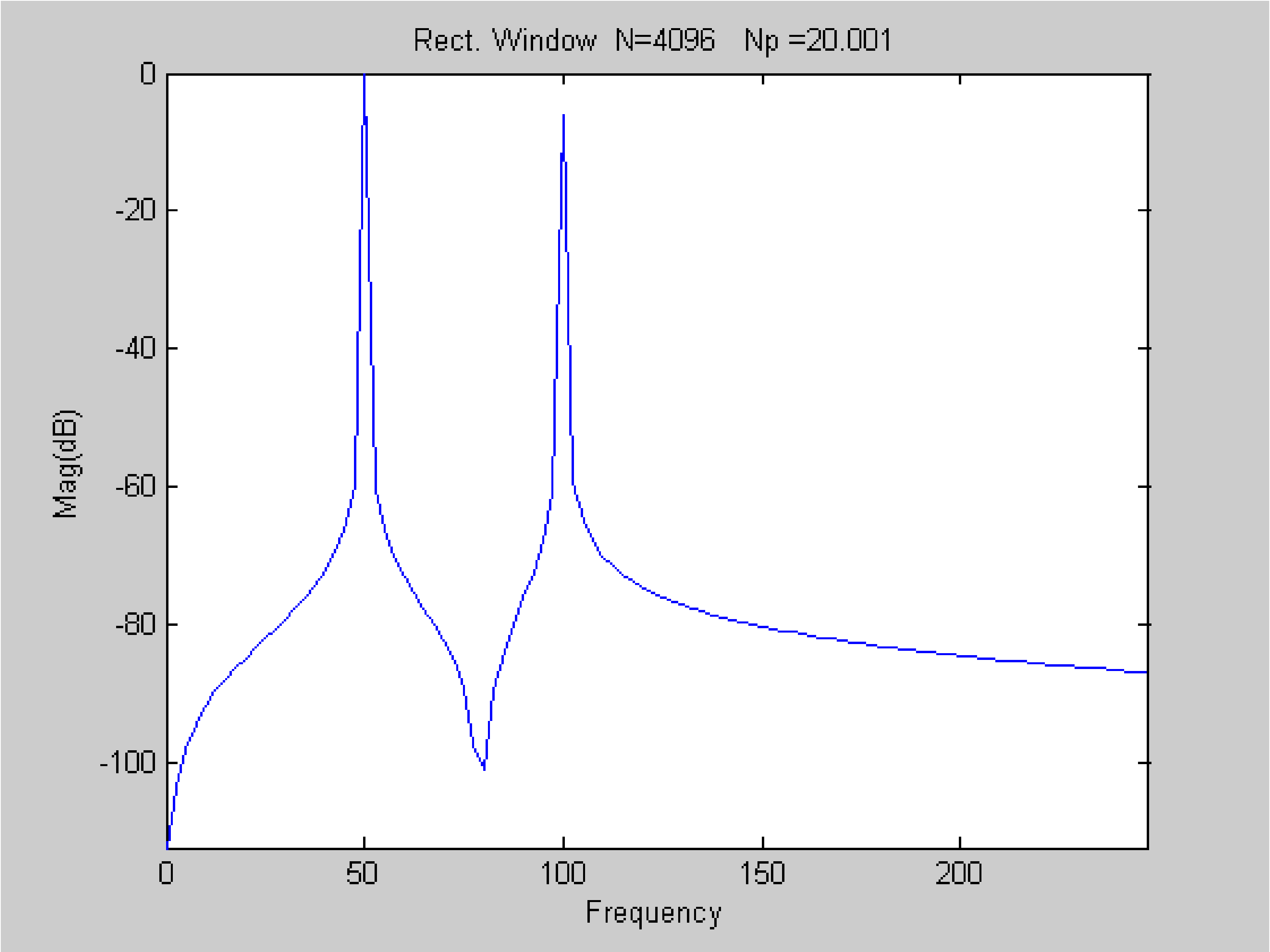
$$V_{IN} = \sin(\omega t) + 0.5\sin(2\omega t)$$

$$\omega = 2\pi f_{SIG}$$

Consider N_P=20.001 N=4096

Deviation from hypothesis is .005% of the sampling window

Spectral Response with Non-coherent Sampling



(zoomed in around fundamental)

Fundamental will appear at position 1+Np = 21

Columns 1 through 7

-112.2531 -103.4507 -97.8283 -94.3021 -91.7015 -89.6024 -87.8059

Columns 8 through 14

-86.2014 -84.7190 -83.3097 -81.9349 -80.5605 -79.1526 -77.6726

Columns 15 through 21

-76.0714 -74.2787 -72.1818 -69.5735 -65.9919 -59.9650 0.0001

Columns 22 through 28

-60.0947 -66.2917 -70.0681 -72.9207 -75.3402 -77.5767 -79.8121

Columns 29 through 35

-82.2405 -85.1651 -89.2710 -97.2462 -101.0487 -89.5195 -83.9851

k^{th} harmonic will appear at position 1+k•Np

Columns 36 through 42

-79.8472 -76.1160 -72.2601 -67.6621 -60.7642 -6.0220 -59.3448

Columns 43 through 49

-64.8177 -67.8520 -69.9156 -71.4625 -72.6918 -73.7078 -74.5718

Columns 50 through 56

-75.3225 -75.9857 -76.5796 -77.1173 -77.6087 -78.0613 -78.4809

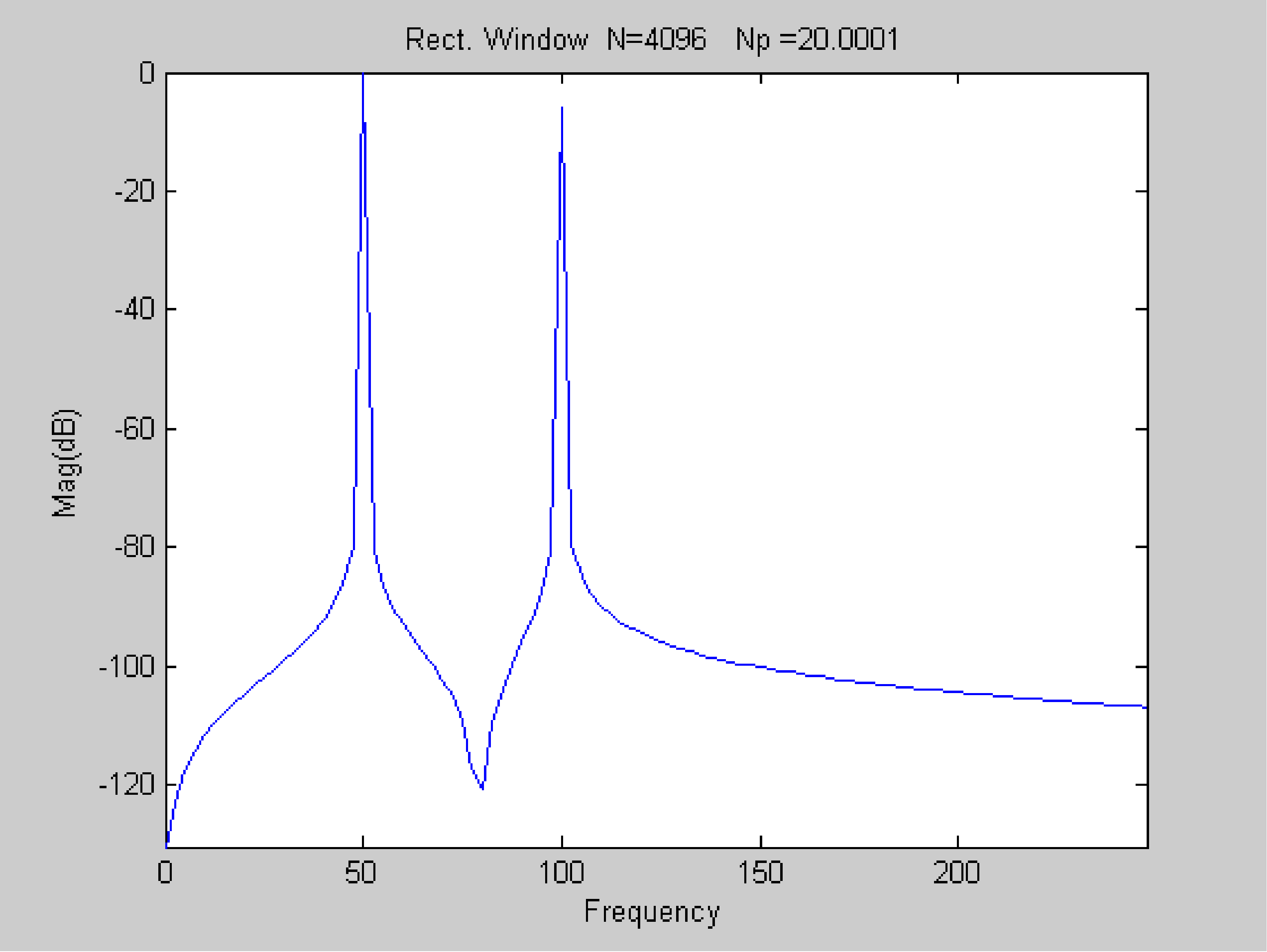
Columns 57 through 63

-78.8721 -79.2387 -79.5837 -79.9096 -80.2186 -80.5125 -80.7927

Observations

- Modest change in sampling window of 0.01 out of 20 periods (.005%) results in a small error in both fundamental and harmonic
- More importantly, substantial raise in the computational noise floor !!! (from over -300dB to only -60dB)
- Errors at about the 10-bit level !

Spectral Response with Non-coherent sampling



(zoomed in around fundamental)

Fundamental will appear at position 1+Np = 21

Columns 1 through 7

-130.4427 -123.1634 -117.7467 -114.2649 -111.6804 -109.5888 -107.7965

Columns 8 through 14

-106.1944 -104.7137 -103.3055 -101.9314 -100.5575 -99.1499 -97.6702

Columns 15 through 21

-96.0691 -94.2764 -92.1793 -89.5706 -85.9878 -79.9571 0.0000

Columns 22 through 28

-80.1027 -86.2959 -90.0712 -92.9232 -95.3425 -97.5788 -99.8141

Columns 29 through 35

-102.2424 -105.1665 -109.2693 -117.2013 -120.8396 -109.4934 -103.9724

k^{th} harmonic will appear at position 1+k•Np

Columns 36 through 42

-99.8382	-96.1082	-92.2521	-87.6522	-80.7470	-6.0207	-79.3595

Columns 43 through 49

-84.8247	-87.8566	-89.9190	-91.4652	-92.6940	-93.7098	-94.5736

Columns 50 through 56

-95.3241	-95.9872	-96.5810	-97.1187	-97.6100	-98.0625	-98.4821

Columns 57 through 63

-98.8732	-99.2398	-99.5847	-99.9107	-100.2197	-100.5135	-100.7937

Columns 64 through 70

Observations

- Modest change in sampling window of 0.001 out of 20 periods (.0005%) results in a small error in both fundamental and harmonic
- More importantly, substantial raise in the computational noise floor !!! (from over -300dB to only -80dB)
- Errors at about the 13-bit level !

Considerations for Spectral Characterization

- Tool Validation
- FFT Length
- Importance of Satisfying Hypothesis
  - NP is an integer
  - Band-limited excitation
- Windowing

Stay Safe and Stay Healthy !

End of Lecture 5